

The maximal existence region of the analytic solutions for a class of characteristic Einstein vacuum equations.

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Abstract

The main goal of this work consists in showing that the analytic solutions for a class of characteristic problems for the Einstein vacuum equations have an existence region larger than the one provided by the Cauchy-Kowalevski theorem, due to the intrinsic hyperbolicity of the Einstein equations. The magnitude of this region depends only on suitable H_s Sobolev norms of the initial data for a fixed $s \leq 7$ and if the initial data are sufficiently small the analytic solution is global. In a previous paper, hereafter “I”, we have described a geometric way of writing the vacuum Einstein equations for the characteristic problems we are considering and a local solution in a suitable “double null cone gauge” characterized by the use of a double null cone foliation of the spacetime. In this paper, using this “geometric” gauge, we give a detailed proof of this extension result.

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1 Introduction

In this paper we prove a result about the existence region of the analytic solutions of a specific class of characteristic problems, namely those whose “initial data” are given on a null hypersurface consisting of the union of a truncated outgoing null cone and of a truncated incoming cone intersecting the previous one along a surface homeomorphic to S^2 . This class of characteristic problems was studied by different authors, for instance H.Muller Zum Hagen, [Mu], H.Muller Zum Hagen and H.J.Seifert, [Mu-Se], in a series of papers by Dossa, see [Do] and references therein, but, in particular, in the anticipating work by A.Rendall [Ren], where a thorough examination is done showing how to obtain initial data satisfying the constraint equations and the harmonic conditions and, subsequently, a local existence result is presented. Recently, following, but largely improving the A.Rendall result, one has to recall the paper by Y. Choquet-Bruhat, P.Y.Chrusciel, J.M. Martin-Garcia “The Cauchy problem on a characteristic cone for the Einstein equations in arbitrary dimensions”, see [Cb-Ch-Do]. There the authors prove a local existence result for the characteristic problem with initial data on a null cone, using again the harmonic gauge and proving in a very detailed way how the initial data constraints have to be satisfied and how, relying on Dossa results, [Do], the local existence result can be proved. Moreover the nature of the characteristic problem with initial data on the null cone, adds the extra problem of the “tip of the cone” which they solve completely.

In the present paper our main goal is to show that the real analytic solutions of the class of characteristic problems we are considering have, due to the hyperbolicity of the Einstein equations, a larger existence region than the one proved by the application of the Cauchy-Kowalevski theorem.

More precisely the results of this paper is twofold; first we prove how to provide analitic initial data, namely which part of them has to be given in a free way on the whole initial hypersurface and which part has to be given only on the intersection of the incoming and outgoing cones and then obtained on the whole initial hypersurface using the constraint equations. Second, we prove that, given analytic initial data on the null cones, the extension of the analyticity existence region depends only on a finite number of derivatives of the initial data, that is on some appropriate H_s Sobolev norms with a given s ; hence if we assume the initial data “small” in these norms, we can prove the global existence for the analytic solutions. Moreover this implies immediately the global existence of weak solutions with small H_s initial data. At the end of the introduction we will state the two theorems summarizing these results.

Some analogous results have been obtained in the past by S.Alinhac, G.Metivier, where they proved the propagation of the analiticity for hyperbolic systems of p.d.e., see [A-M] and references therein;

In most of the proofs of the global existence a bootstrap argument is exploited: one assumes the spacetime foliated by a family of spacelike hypersurfaces, which amounts basically to choose a gauge, and uses the evolution equations written in this gauge to control the metric restricted to these hypersurfaces and its deriva-

tives up to a certain order, proving that the solution can be extended beyond the “last” hypersurface and, by contradiction, the global existence. Nevertheless when we have to deal with a global existence problem it turns out very problematic, using a foliation made by spacelike hypersurfaces, to obtain a priori energy-type estimates which can be bounded globally.¹

Following the seminal ideas of Christodoulou and Klainerman, [Ch-Kl], it turns out that in many instances it appears more appropriate to use a foliation made by a family of null hypersurfaces (generally called cones), outgoing in that case and both outgoing and incoming in [Kl-Ni2]. It was in the work of Christodoulou and Klainerman, [Ch-Kl], that the crucial use of the foliation in terms of the outgoing cones allowed to prove that the energy norms, denoted in general with \mathcal{Q} , are bounded everywhere.

The choice of this foliation, we call “double null canonical gauge”, completely specified later on, is even more appropriate in the case of characteristic problems where also the “initial data” are given on null hypersurfaces. Therefore this choice is the first step needed to face and solve our problem.

These aspects of our result, the construction of the foliation, the explicit expression which the Einstein equations assume, the precise separation between the constraint and the evolution equations and the general strategy to construct the analytic solution and prove its extension to a region larger than the one provided by the Cauchy-Kowalevski theorem are the issues carefully discussed in (I). The main conclusion proved there is that, assuming the initial data belonging to a function space denoted $\mathcal{B}_{\alpha, \rho_0}$ ² defined in (I)³, to prove that the Cauchy Kowalevski solution region can be extended, we have to prove that in this region the solution belongs to a function space $\mathcal{B}_{\alpha, \rho}$ with $\rho < \rho_0$ depending only on the first derivatives norms, that is to an H_s norms of the solution, with $s \leq J_0 < \infty$.⁴ Moreover if these first derivatives norms are sufficiently small everywhere then the solution is globally analytic. To prove it we need, as said before, that the initial data are small. The proof of this result is the core of this work.

¹See for instance the Linblad Rodnianski paper, [Lin-Rod], where the “energy norms” are not bounded, but increase in t .

²See 1.3 for the definition of such space, observe that in (I) $\mathcal{B}_{\alpha, \rho}$ is defined with the partial derivatives while here with the covariant ones.

³Notice that if the initial data belong to $\mathcal{B}_{\alpha, \rho_0}$ then they are analytic with radius ρ_0 viceversa if they are analytic with radius ρ_* then they belong to $\mathcal{B}_{\alpha, \rho_0}$ for suitable $\rho_0 < \rho_*$, see lemmas 4.1 and 4.2 of (I).

⁴In the following we choose $J_0 = 7$, the precise reason for this choice is discussed in [Ca-Ni1].

In the remaining part of this introduction we give a survey of all the main technical steps needed to prove this result, which requires the control of all order derivatives of the solution components on the leaves of the “double null canonical foliation”.

1) The gauge: This first step was the main content of (I). We just recall, as we said before, that the choice of this foliation made by null outgoing and incoming hypersurfaces (cones) allows to define a natural set of coordinates $\{\nu, \lambda, \theta, \phi\}$ and look for the Einstein equations solutions as a set of analytic functions in these coordinates. The precise definition of all the quantities defining a solution of the characteristic Einstein problem and the explicit expression of the Einstein equations in this gauge are given in section 2. Here we simply recall that, as discussed at length in (I), the metric of the spacetime has, in this gauge, the following expression,⁵

$$\mathbf{g} = -2\Omega^2(dud\underline{u} + d\underline{u}du) + \gamma_{ab}(X^a du + d\omega^a)(X^b du + d\omega^b) , \quad (1.1)$$

where γ_{ab} is the metric induced on the two-dimensional hypersurface $S(\lambda, \nu)$, intersection of the outgoing cone $C(\lambda)$ and the incoming one $\underline{C}(\nu)$

$$S(\lambda, \nu) = C(\lambda) \cap \underline{C}(\nu) ,$$

λ, ν are the coordinates along the incoming and outgoing cones respectively, Ω has the role of the lapse function in the gauge associated to the more standard spacelike foliations and X is the analogous of the shift vector and accounts to the non spherical symmetric of the spacetime. The initial data null cones will be indicated as $C_0 = C(\lambda_0)$, for the outgoing one and $\underline{C}_0 = \underline{C}(\nu_0)$ for the incoming one. Finally to each point of the spacetime a null orthonormal frame can be associated, $\{e_3, e_4, e_\theta, e_\phi\}$ with e_3 and e_4 null vectors, $\mathbf{g}(e_3, e_4) = -2$ and the equivariant vector fields⁶

$$N = \Omega e_4 = \frac{\partial}{\partial \nu} \quad ; \quad \underline{N} = \Omega e_3 = \frac{\partial}{\partial \lambda} + X^a \frac{\partial}{\partial \omega^a} \quad (1.2)$$

where $\{\omega^a\} = \{\theta, \phi\}$.

The general solution, denoted by Φ , see (I), is made by various terms,

$$\Phi = (\mathbf{g}; \mathcal{O}, \underline{\mathcal{O}}) \equiv (\mathbf{g}; \mathcal{U}) = (\gamma, \Omega, v, \psi, w, X; \omega, \zeta, \chi, \underline{\omega}, \underline{\chi}) .$$

where $\gamma, \Omega, v, \psi, w, X$ are the metric components or their first angular derivatives, while $\omega, \zeta, \chi, \underline{\omega}, \underline{\chi}$ are the connection coefficients which are, basically, expressions of the remaining first derivatives of the metric components,

$$\begin{aligned} \chi_{AB} &= \mathbf{g}(\mathbf{D}_{e_A} e_4, e_B) \quad , \quad \underline{\chi}_{AB} = \mathbf{g}(\mathbf{D}_{e_A} e_3, e_B) \\ \underline{\xi}_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_{e_3} e_3, e_A) \quad , \quad \xi_A = \frac{1}{2} \mathbf{g}(\mathbf{D}_{e_4} e_4, e_A) \end{aligned}$$

⁵Here and in the following we denote in an interchangeable way u, \underline{u} and λ, ν , the values they take, to denote the null coordinates.

⁶See (I) for their main properties.

$$\begin{aligned}
\underline{\omega} &= \frac{1}{4} \mathbf{g}(\mathbf{D}_{e_3} e_3, e_4) \quad , \quad \omega = \frac{1}{4} \mathbf{g}(\mathbf{D}_{e_4} e_4, e_3) \\
\underline{\eta}_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_{e_4} e_3, e_A) \quad , \quad \eta_A = \frac{1}{2} \mathbf{g}(\mathbf{D}_{e_3} e_4, e_A) \\
\zeta_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_{e_A} e_4, e_3) \quad .
\end{aligned}$$

Moreover we define the initial data set of the metric components and connection coefficients:

$$\Phi^{(0)} = (\mathbf{g}; \mathcal{O}, \underline{\mathcal{Q}})|_{C_0 \cup \underline{C}_0}.$$

2) The hierarchy: As discussed at length in (I) we have to show that our analytic solution defined in the region \mathcal{K} is in a $\mathcal{B}_{\alpha, \rho}$ function space with a $\rho < \rho_0$ depending only on the first derivatives norms of the solution. More precisely we prove the following bounds for all the cone tangential derivatives of the solution components, we denote generically by \mathcal{V} ,⁷

$$|r^{2+J-\frac{2}{p}} \nabla^J \mathcal{V}|_{p, S}(\lambda, \nu) \leq C_0 \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\underline{\Gamma}(\lambda))}}{\rho^J} . \quad (1.3)$$

The $|\cdot|_{p, S}$ norms are the norms we will use systematically. They are defined as the L_p norms on the surface $S(\lambda, \nu)$ endowed with the induced metric, with $p \in [2, 4]$, ∇ denotes the covariant tangential derivatives⁸, by these norms we define in the obvious way also the H_s spaces⁹. On the right hand side, $\delta > 0$ has to be chosen sufficiently large and $\underline{\Gamma}(\lambda)$ is defined in the following section. To prove it we have to control in a appropriate way the norms of all the tangential derivatives of these components along the outgoing and the incoming cones; therefore along the outgoing cones we have to control the angular derivatives and the derivatives with respect to e_4 , while for the incoming cones the angular ones and the derivatives with respect to e_3 . Once these bounds are proved it is immediate to show that the solution belongs to the $\mathcal{B}_{\alpha, \hat{\rho}}$ defined in (I) (with the ordinary partial derivatives) with $\hat{\rho} < \rho$ and, therefore, to obtain the desired result.

The core of this paper is, therefore, to prove that the estimates 1.3 hold with ρ depending only on the first 7 covariant derivatives. This has to be done in various steps, following a precise order in the estimates of the various Φ components, which shows a hierarchy in the technical results we need.

We start proving the estimates 1.3 for the connection coefficients part of the solution Φ , namely $\mathcal{U} \equiv \{\chi, \zeta, \omega, \underline{\chi}, \underline{\omega}\}$ and more precisely we prove these estimates for their covariant angular derivatives, ∇ , see Theorem 3.1. This is the

⁷The estimates for $\gamma, \Omega, v, \psi, w, X$ are slightly different and their precise estimates are given later on.

⁸It is well known that the definition of tangential derivatives along the null cones is in some sense ambiguous. Here with tangential with respect to the outgoing cone we mean, besides the angular derivatives those along e_4 and e_3 for the incoming one.

⁹notice that it is equivalent to require the smallness in H_s or to require that the constant in front of the right hand side of equation 1.3 is small for the first s derivatives.

first and central result and the largest part of this work; to get it we have to use in a substantial way those structure equations which have the form of transport equations without any “loss of derivatives”¹⁰; once these estimates are obtained solving these transport equations we can obtain the estimates for all the remaining tangential derivatives in a simpler way using the structure equations without worrying anymore on the possible “loss of derivatives” as we already control all order angular derivatives. Finally once we control the tangential derivatives of the connection coefficients we obtain from them the non covariant derivatives of the same quantities completing the estimates we need to prove that the solution belongs to $\mathcal{B}_{\alpha,\hat{\rho}}$.

3) The extension:

As we proved in (I), once the solution belongs to $\mathcal{B}_{\alpha,\hat{\rho}}$ in the whole region \mathcal{K} , we assume being the largest analyticity region, we can use these estimates on the upper boundary $\partial\mathcal{K}_+$ of \mathcal{K} , to extend, via Cauchy-Kowalewski, the analytic solution in a small strip above it. This implies that \mathcal{K} is not the largest possible analyticity region and, therefore, the solution is global. It is important to remark that to claim that the solution is global, we need that the a priori estimates for the energy-type quantities associated to the H^s solutions with $s \leq 7$, hold in the whole spacetime. This requirement is satisfied if the first 7 derivatives initial data are sufficiently small. If viceversa these initial data are not small, these a priori estimates hold only in a finite region, $\bar{\mathcal{K}}$, smaller as the initial data are larger and in this case the analyticity region \mathcal{K} cannot be extended beyond $\bar{\mathcal{K}}$.

Due to the complexity of this work we give in the rest of this section some more details of all the main steps quoted before, which will be treated exhaustively in the subsequent sections.

The angular derivatives: As we said, to estimate the covariant angular derivatives of the connection coefficients we use those structure equations which have the form of transport equations. To control the not underlined connection coefficients we use the transport equations along the outgoing cones, while for the underlined quantities those along the incoming cones.¹¹ The way to obtain the norm estimates is an inductive one, we assume to control the angular derivatives norms up to order $J \leq N - 1$ ¹² and we prove the analogous estimates for $J = N$. This procedure to be satisfied requires, therefore, the use of those transport equations where no “loss of derivatives” is present. There are three important aspects which have to be emphasized here as they are deeply entangled with the mathematical structure of the equations we are solving.

i) A clear example of the first aspect, present in all the situations when trying to solve the transport equation for the connection coefficients, appears when we look for the estimate of the trace part of χ , $\text{tr}\chi$, and its angular derivatives. Its

¹⁰see in (I) eqs. (2.39).

¹¹With the exception of ω and $\underline{\omega}$, see later.

¹²To be precise, for technical reasons, we will assume inductive estimates for the \mathcal{L}_O^J angular derivatives of the connection coefficients.

transport equation, written for $U = \Omega^{-1}\text{tr}\chi$,¹³

$$\frac{\partial U}{\partial \nu} + \frac{\Omega \text{tr}\chi}{2} U + |\hat{\chi}|^2 = 0, \quad (1.4)$$

depends on the traceless part, $\hat{\chi}$, of χ and, looking at the structure equations, see for instance (I) subsection 6.1.1, it is immediate to realize that there is not a transport equation for $\hat{\chi}$ without a loss of derivatives.

To overcome this problem we remark that the structure equations provide us also with some elliptic Hodge equations defined on the generic two dimensional surface, S , intersection of an outgoing cone and an incoming cone of the foliation. In particular for $\hat{\chi}$ we have the following elliptic equations

$$\text{div} \hat{\chi} = \frac{1}{2} \nabla \text{tr}\chi - \beta - \zeta \cdot \hat{\chi} + \frac{1}{2} \zeta \text{tr}\chi \quad (1.5)$$

Nevertheless to use this equation to control $\hat{\chi}$ in 1.4 we have to control the null Riemann components, in the present case β . This is where the hyperbolic nature of the Einstein equations plays its important role. In fact to treat 1.5 as an equation and not as the definition of the Riemann component β we need an independent way to control the null Riemann components.

ii) The hyperbolic structure is the second important aspect, provided by the hyperbolic nature of the Einstein equations which allows to obtain some apriori estimates for some energy-type quantities, L^2 norms of the Bel-Robinson tensor and suitable Lie derivatives, denoted hereafter in general with \mathcal{Q} and defined in any detail in Section 4.

These “a priori energy estimates” allow to control the Riemann components, in the previous case β , in terms of the analogous initial data quantities, therefore, to control via equations 1.5 $\hat{\chi}$ and its angular derivatives and, finally, through the transport equation 1.4, $\text{tr}\chi$. Clearly these estimates need, to be performed, the boundedness of the derivatives of the connection coefficients and it is here that the bootstrap mechanism enter. The mechanism just sketched, discussed in greater detail in subsection 3.2, reproduces itself, with some technical, but not substantial, differences to control all the remaining connection coefficients and their angular derivatives. It is the basic use of these a priori estimates which implies that the analytic solution can be extended up to the region where these a priori estimates hold.

iii) The third aspect to emphasize is that looking at the transport equations, one realizes that to control, via the inductive mechanism, all the angular derivatives of the connection coefficients we need to control the asymptotic decay of these quantities along the outgoing cones. This requires, for technical reasons, that we have to integrate the outgoing cones transport equations from above and that, as discussed in any detail in subsections 9.5, 9.6.1 and in subsection 14.11, the foliation of the spacetime has to be chosen in an appropriate way. This is the reason of the introduction of the so called “double null cone canonical

¹³The advantage to look at U instead of $\text{tr}\chi$ will be clarified in the next section.

foliation” which, beside its technical need to prove our result, basically satisfies the request that the outgoing cones should coincide asymptotically with those of the Schwarzschild spacetime.

The mixed derivatives and the metric components derivatives: As said before discussing the natural hierarchy of the steps required to prove the result, once we control the angular derivatives of the connection coefficients, the subsequent step is to control the $\mathbf{D}_4, \mathbf{D}_3$ derivatives of all order and the mixed ones with the angular derivatives. This will require to introduce the transport equations for the $\mathcal{L}_S^p \mathcal{L}_O^q$, derivatives of the connection coefficients and to define a class of extended energy norms, in order to repeat all the steps performed for the \mathcal{L}_O^q derivatives for the $\mathcal{L}_S^p \mathcal{L}_O^q$ ones. The last step will be to obtain estimates for the “not ordered” mixed derivatives. A different way of obtaining the mixed derivatives estimates can be performed, see the remark after Theorem 3.9, emphasizing the central role of the angular derivatives.

Once we control all the covariant derivatives of the connection coefficients using equations similar to 1.4 and 1.5 it is a long, but easy procedure to obtain the estimates for the metric components; this is the content of Section 5.

The initial data: Finally, in Section 7, we present a careful discussion of how obtaining the class of (global) initial data required to have global analytic solutions. While here we present the explicit way to construct initial data on the whole (unbounded) initial data null hypersurfaces, we refer to (I) for the general discussion of how, in our double null canonical gauge, the constraints appear and have to be satisfied.

We end this introduction stating two theorems summarizing our results; they are given here in a compact form and the following sections are devoted to state their precise versions and their proofs.¹⁴

Theorem 1.1 *Let us assigne $\hat{\chi}$ on C_0 , $\hat{\underline{\chi}}$ and X on \underline{C}_0 and Ω on $C_0 \cup \underline{C}_0$ analytic in $\mathcal{B}_{\alpha; \rho_0, 0, 1}$, moreover assume assigned¹⁵ $\{\gamma, \zeta, tr\chi, tr\underline{\chi}, \underline{\omega}, \mathbf{K}\}$, analytic and satisfying suitable equations on S_0 . If all these coefficients satisfy suitable smallness bounds in $|\cdot|_{p, S}$ norms together with their first 7 tangential derivatives, then it is possible to construct analytic initial data Φ^0 on $C_0 \cup \underline{C}_0$ in $\mathcal{B}_{\alpha; \rho_0}$, with $\alpha > 3$, $\rho_0 < \rho_{0, 0, 1}$ and the energy type \mathcal{Q}^0 norms¹⁶ on $C_0 \cup \underline{C}_0$ finite and small¹⁷.*

Theorem 1.2 *Let us give Φ , a real analytic solution of the characteristic problem for the Einstein vacuum equations in the region*

$$\mathcal{K}(\Lambda_a, \Pi_a) \equiv \{(\lambda, \nu) \in [\lambda_0, \Lambda_a] \times [\nu_0, \Pi_a]\} ,$$

domain of dependance of $C_0(\lambda, [\nu_0, \Pi_a]) \cup \underline{C}_0([\Lambda_0, \lambda_0], \nu)$, with initial data set Φ^0 belonging to the Banach space $\mathcal{B}_{\alpha; \rho_0}$, with $\alpha > 3$. Then there exists a double

¹⁴see theorem 3.2

¹⁵ \mathbf{K} is the curvature of S_0

¹⁶see definition 4.2

¹⁷the precise version of this theorem is in section 7

null foliation, the “double null canonical foliation”,¹⁸ of the region $\mathcal{K}(\Lambda_a, \Pi_a)$ such that the generic connection coefficient related to Φ and associated with such foliation, belong to $\mathcal{B}_{\alpha, \hat{\rho}}$, with $\hat{\rho} < \rho_0$ depending only on the $|\cdot|_{p,S}$ norms of the first s derivatives, $s \leq 7$, of the initial data Φ^0 . Moreover if we give Φ^0 such that the energy type \mathcal{Q}^0 norms on $C_0 \cup \underline{C}_0$ are finite and sufficiently small, we can construct an analytic solution in all the domain of dependence of $C_0 \cup \underline{C}_0$ belonging to $\mathcal{B}_{\alpha, \hat{\rho}}$.

From these two theorems the following corollary can be derived:

Corollary 1.3 *Given $\hat{\chi}$ on C_0 , $\hat{\chi}$ and X on \underline{C}_0 and Ω on $C_0 \cup \underline{C}_0$, satisfying suitable smallness bounds in $|\cdot|_{p,S}$ norms with their first s derivatives, $s \leq 7$, and $\{\gamma, \zeta, \text{tr}\chi, \text{tr}\underline{\chi}, \underline{\omega}, \mathbf{K}\}$ on S_0 , satisfying suitable equations and small in $|\cdot|_{p,S_0}$, with their first s angular derivatives, we can construct a weak global solution, in $|\cdot|_{p,S}$ norms with their first s derivatives, $s \leq 7$, in the whole domain of dependence of $C_0 \cup \underline{C}_0$.*

Remark:

Notice that in the two theorems and the corollary, the smallness conditions for the initial data are not precisely specified, this is because they are related to the smallness of the \mathcal{Q}^0 energy norms, this smallness, on the other hand, is determined in a precise but not straightforward way in [Kl-Ni2], chapter 6. These to types of smallness are intimately related, hence we refer to them generally with $O(\epsilon)$.

Acknowledgments: The idea propagate as much as possible the analiticity of the solutions of the Einstein equations was suggested, years ago, to one of the author (F.N.) by S.Klainerman and was implemented together on the “toy model” of the Burger equation in [Kl-Ni1]; moreover both present authors are pleased to thank S.Klainerman one for the long and rich collaboration and (G.C.) for the useful period spent in the Math. Department of the Princeton University and for the long and profitable conversations he had with him.

¹⁸Its explicit definition is in [Kl-Ni2], Chapter 3, see also [Ch-Kl] where it was first introduced. Its definition and some specific details for the present case are given in subsections 9.5 and 14.11.

2 The Einstein equations in the double null canonical gauge

We collect here some of the results proved in (I) to make this work more complete and to introduce all the notations systematically used in the following. In the chosen gauge, see (I) for details, the evolution part of the Einstein equations is made by the following system of first order equations for the various tensors components omitting the indices to simplify the notations,¹⁹

$$\begin{aligned}
& \frac{\partial \gamma}{\partial \omega} - v = 0 \quad , \quad \frac{\partial \log \Omega}{\partial \omega} - \psi = 0 \quad , \quad \frac{\partial \hat{X}}{\partial \omega} - w = 0 \\
& \frac{\partial \gamma}{\partial \lambda} - 2\Omega \underline{\chi} + \mathcal{L}_X \gamma = 0 \\
& \frac{\partial \log \Omega}{\partial \lambda} + \psi(X) + 2\Omega \underline{\omega} = 0 \\
& \frac{\partial v}{\partial \lambda} + \nabla_X v + (\not\partial X) \cdot v - S(\not\partial \otimes w) - 2\Omega \not\partial \otimes \underline{\chi} - 2\Omega \psi \otimes \underline{\chi} = 0 \\
& \frac{\partial \psi}{\partial \lambda} + \nabla_X \psi + 2\Omega \underline{\omega} \psi + \psi(\nabla X) + 2\Omega \not\partial \underline{\omega} = 0 \\
& \frac{\partial \text{tr} \chi}{\partial \lambda} + \Omega \text{tr} \chi \text{tr} \chi - 2\Omega \underline{\omega} \text{tr} \chi + \nabla_X \text{tr} \chi - 2\Omega \text{div}(\zeta + \psi) - 2\Omega |\zeta + \psi|^2 + 2\Omega \mathbf{K} = 0 \\
& \frac{\partial \hat{\chi}}{\partial \lambda} + \mathcal{L}_X \hat{\chi} - \frac{\Omega \text{tr} \chi}{2} \hat{\chi} + \frac{\Omega \text{tr} \chi}{2} \underline{\hat{\chi}} - 2\Omega \underline{\omega} \hat{\chi} - \Omega(\hat{\chi} \cdot \underline{\hat{\chi}}) \gamma - \Omega \nabla \hat{\otimes}(\zeta + \psi) - \Omega(\zeta + \psi) \hat{\otimes}(\zeta + \psi) = 0 \\
& \frac{\partial \zeta}{\partial \lambda} + \mathcal{L}_X \zeta + \Omega \text{tr} \chi \zeta + \Omega \text{div} \underline{\hat{\chi}} - \frac{1}{2} \Omega \not\partial \text{tr} \chi + 2\Omega \underline{\omega} \psi + 2\Omega \not\partial \underline{\omega} + \Omega \psi \cdot \underline{\chi} = 0 \\
& \frac{\partial \omega}{\partial \lambda} + \not\partial_X \omega - 2\Omega \underline{\omega} \omega - \frac{3}{2} \Omega |\zeta|^2 + \frac{1}{4} \Omega \zeta \cdot \psi + \frac{1}{2} \Omega |\psi|^2 + \frac{1}{2} \Omega \left(\mathbf{K} + \frac{1}{4} \text{tr} \chi \text{tr} \chi - \frac{1}{2} \hat{\chi} \cdot \underline{\hat{\chi}} \right) = 0 \\
& \frac{\partial \hat{X}}{\partial \nu} + 4\Omega^2 \zeta = 0 \\
& \frac{\partial w}{\partial \nu} + 8\Omega^2 \psi \otimes \zeta + 4\Omega^2 \not\partial \otimes \zeta - 2\Omega \psi \otimes (\chi \cdot X) - 2\Omega (\not\partial \otimes \chi) \cdot X = 0 \\
& \frac{\partial \text{tr} \chi}{\partial \nu} + \Omega \text{tr} \chi \text{tr} \chi - 2\Omega \omega \text{tr} \chi + 2\Omega \text{div} \zeta - 2\Omega \text{div} \psi - 2\Omega |\zeta - \psi|^2 + 2\Omega \mathbf{K} = 0 \\
& \frac{\partial \underline{\hat{\chi}}}{\partial \nu} - \frac{\Omega \text{tr} \chi}{2} \underline{\hat{\chi}} + \frac{\Omega \text{tr} \chi}{2} \hat{\chi} - 2\Omega \omega \underline{\hat{\chi}} - \Omega(\underline{\hat{\chi}} \cdot \hat{\chi}) \gamma + \Omega \nabla \hat{\otimes}(\zeta - \psi) - \Omega(\zeta - \psi) \hat{\otimes}(\zeta - \psi) = 0 \\
& \frac{\partial \underline{\omega}}{\partial \nu} - 2\Omega \omega \underline{\omega} - \frac{3}{2} \Omega |\zeta|^2 - \Omega \zeta \cdot \psi + \frac{1}{2} \Omega |\psi|^2 + \frac{1}{2} \Omega \left(\mathbf{K} + \frac{1}{4} \text{tr} \chi \text{tr} \chi - \frac{1}{2} \hat{\chi} \cdot \underline{\hat{\chi}} \right) = 0 \quad ,
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
& \frac{\partial \hat{X}}{\partial \nu} + 4\Omega^2 \zeta = 0 \\
& \frac{\partial w}{\partial \nu} + 8\Omega^2 \psi \otimes \zeta + 4\Omega^2 \not\partial \otimes \zeta - 2\Omega \psi \otimes (\chi \cdot X) - 2\Omega (\not\partial \otimes \chi) \cdot X = 0 \\
& \frac{\partial \text{tr} \chi}{\partial \nu} + \Omega \text{tr} \chi \text{tr} \chi - 2\Omega \omega \text{tr} \chi + 2\Omega \text{div} \zeta - 2\Omega \text{div} \psi - 2\Omega |\zeta - \psi|^2 + 2\Omega \mathbf{K} = 0 \\
& \frac{\partial \underline{\hat{\chi}}}{\partial \nu} - \frac{\Omega \text{tr} \chi}{2} \underline{\hat{\chi}} + \frac{\Omega \text{tr} \chi}{2} \hat{\chi} - 2\Omega \omega \underline{\hat{\chi}} - \Omega(\underline{\hat{\chi}} \cdot \hat{\chi}) \gamma + \Omega \nabla \hat{\otimes}(\zeta - \psi) - \Omega(\zeta - \psi) \hat{\otimes}(\zeta - \psi) = 0 \\
& \frac{\partial \underline{\omega}}{\partial \nu} - 2\Omega \omega \underline{\omega} - \frac{3}{2} \Omega |\zeta|^2 - \Omega \zeta \cdot \psi + \frac{1}{2} \Omega |\psi|^2 + \frac{1}{2} \Omega \left(\mathbf{K} + \frac{1}{4} \text{tr} \chi \text{tr} \chi - \frac{1}{2} \hat{\chi} \cdot \underline{\hat{\chi}} \right) = 0 \quad ,
\end{aligned} \tag{2.2}$$

where γ, Ω, X are the components of the metric which has the following expression

$$\mathbf{g} = -2\Omega^2(dud\underline{u} + d\underline{u}du) + \gamma_{ab}(X^a du + d\omega^a)(X^b du + d\omega^b) \quad . \tag{2.3}$$

¹⁹In the first line of equations ω denotes the angular coordinates.

The solution of the characteristic problem specified by the equations 2.1 and 2.2 becomes a solution of the Einstein equations if the initial data satisfy the following equations, which have to be interpreted as constraint equations,²⁰

$$\begin{aligned}
\text{On } C_0 : \quad & \frac{\partial \gamma}{\partial \nu} - 2\Omega\chi = 0 \quad , \quad \frac{\partial \log \Omega}{\partial \nu} + 2\Omega\omega = 0 \\
& \frac{\partial \text{tr} \chi}{\partial \nu} + \frac{\Omega \text{tr} \chi}{2} \text{tr} \chi + 2\Omega\omega \text{tr} \chi + \Omega|\hat{\chi}|^2 = 0 \\
& \frac{\partial \zeta}{\partial \nu} + \Omega \text{tr} \chi \zeta - \Omega \text{div} \chi + \Omega \not{\partial} \text{tr} \chi + \frac{\partial \not{\partial} \log \Omega}{\partial \nu} - \Omega \not{\partial} \log \Omega \cdot \chi = 0 \\
\\
\text{On } \underline{C}_0 : \quad & \frac{\partial \text{tr} \underline{\chi}}{\partial \underline{\lambda}} + \frac{\Omega \text{tr} \underline{\chi}}{2} \text{tr} \underline{\chi} + \partial_{\underline{\chi}} \text{tr} \underline{\chi} + 2\Omega \underline{\omega} \text{tr} \underline{\chi} + \Omega|\hat{\underline{\chi}}|^2 = 0 \quad . \quad (2.4)
\end{aligned}$$

In the next sections we start proving all the technical lemmas and theorems needed to obtain the final result, following the order required by the hierarchical structure described in the introduction.

3 The covariant angular derivatives for the connection coefficients

As said in the introduction, we have to estimate first the angular derivatives of the connection coefficients. The main result of this section is summarized in the following theorem, which was stated in (I) as Lemma (I;4.7). To prove it we assume the apriori estimates for some energy type norms, associated to the Riemann tensor, which are the main content of the hyperbolicity of the Einstein equations and which are proved in Section 4 adapting to the present problem the results proved in [Kl-Ni2], Chapter 6.²¹

Theorem 3.1 *Let $\Phi = (\mathbf{g}; \mathcal{O}, \mathcal{Q}) = (\mathbf{g}; \mathcal{U}) = (\gamma, \Omega, v, \psi, w, X; \omega, \zeta, \chi, \underline{\omega}, \underline{\chi})$ be a real analytic solution in the region,²²*

$$\mathcal{K}(\Lambda_a, \Pi_a) \equiv \{(\lambda, \nu) \in [\lambda_0, \Lambda_a] \times [\nu_0, \Pi_a]\} \quad ,$$

of the characteristic problem for the Einstein vacuum equations with analytic initial data, $\{\Phi^{(0)}\}$, on $C_0 \cup \underline{C}_0$ belonging to the Banach space $\mathcal{B}_{\alpha; \rho_0}$, with $\alpha > 3$.

²⁰See (I), Theorem 2.1, for a detailed discussion of this aspect.

²¹Reminding the analogy discussed in (I) of this problem with the much easier one associated to the Burger equation we observe that here some important differences show up with the way we treat the Burger case. There the apriori estimate provided from Lemma (I;4.4) can be immediately derived as a straightforward consequence of the hyperbolicity of the Burger equation while in the case of the Einstein equations their hyperbolic nature is somewhat hidden unless we do use some appropriate gauges. Therefore in the present case the order of the proofs is inverted, we prove first Theorem (I;4.6) where we use the Riemann component estimates; subsequently we prove the Riemann estimates.

²²See (I) for its precise definition.

Then there exists a double null foliation, the “double null canonical foliation”,²³ of the region $\mathcal{K}(\Lambda_a, \Pi_a)$ such that the generic connection coefficient we indicate with \mathcal{V} , satisfy the following estimates²⁴

$$|r^{2+J-\frac{2}{p}} \nabla^J \mathcal{V}|_{p,S}(\lambda, \nu) \leq C_1 \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\underline{\Gamma}(\lambda))}}{\rho^J} \quad (3.1)$$

with $\rho < \rho_0$ depending only on ρ_0 and on $\|\Phi^{(0)}\|_J \equiv |\nabla^J \Phi^{(0)}|_{p,S}$ ²⁵ with $J \leq 7$, with C_1 a suitable constant satisfying,

$$C_1 > \|\Phi^{(0)}\|_{B_{\alpha, \rho_0}}, \quad (3.2)$$

and²⁶

$$\underline{\Gamma}(\lambda) = \hat{C} \frac{(\lambda - \lambda_0)}{\lambda \lambda_0}. \quad (3.3)$$

For suitable values of δ and \hat{C} .

Remarks:

i) As said above and discussed in more detail later on, the constant C_1 in 3.1 has not to be small; the situation is different for $J \leq 7$ as in this case, the constant have to be assumed small, $O(\varepsilon)$ ²⁷, if the solution has to be global. Otherwise if we do not ask this “smallness” the analytic solution will exist only in the region where the a priori energy estimates are satisfied, see also the remark at the end of subsection 3.4.

ii) To complete our proof we have first to obtain estimates analogous to 3.1 for the \mathcal{L}_O^N derivatives²⁸ of the connection coefficients instead of the ∇^N ones. To do this we have to exploit the structure equations for $\mathcal{L}_O^N \mathcal{O}$ quantities. The reason for this intermediate step lies in the bootstrap argument we have to use to in this approach. In fact, as said before, to estimate those connection coefficients whose structure equations depends on the null Riemann components, we

²³Its explicit definition is in [Kl-Ni2], Chapter 3, see also [Ch-Kl] where it was first introduced. Its definition and some specific details for the present case are given in subsections 9.5 and 14.11 .

²⁴The notations in inequality 3.1 are a bit symbolic, J is an integer except that in ∇^J where it has to be considered a multiindex with $|J| = J$, moreover if with \mathcal{V} we denote a \mathcal{Q} connection coefficient then the weight factor $r^{2+J-\frac{2}{p}}$ has to be modified, see equations (I;4.52). Finally for some of the connection coefficients the constant C_1 has to be substituted by $C_1 \log J$, the precise version of this result is in subsection 3.5, Theorem 3.6.

²⁵To prove this theorem we only need the $|\cdot|_{p,S}$ norms of the angular derivatives of the initial data, the null direction will be important for the estimates of the mixed derivatives.

²⁶It is appropriate to remark that \hat{C} has the dimension of a length, $[\hat{C}] = L^1$.

²⁷As said before, the explicit estimate of the constant ε is not easy to give. It have to be chosen such that the energy norm, \mathcal{Q}^0 , defined in section 4 and before in [Kl-Ni2], chapter 6, is sufficiently small.

²⁸With the symbol \mathcal{L}_O we mean the sum of the Lie derivatives with respect to the three ⁽ⁱ⁾ \mathcal{O} rotation vectors.

have to use the energy estimates, (depending on the \mathcal{L}_O derivatives of the null Riemann components) which to be satisfied, require lower order \mathcal{L}_O estimates of connection coefficients. At the end we will recover the estimates for the ∇ derivatives lowering the radius of convergence.

3.1 The proof of Theorem 3.1

The estimates we prove refer to the region $\mathcal{K}(\Lambda_a, \Pi_a)$ where we have an analytic solution Φ of the Einstein equations.²⁹ Therefore in this region all the structure equations can be used, both those along the outgoing cones and those along the incoming ones.³⁰ The estimates of these norms follow the same pattern used in [Kl-Ni2], Chapter 4, the main difference being that now we have to control all order derivatives. This is done in a recursive way, assuming we control the lower derivatives we show that we can control the higher ones. This implies, again generalising [Kl-Ni2], that to obtain this result we need to use those transport equations which do not lose regularity.

This proof is very long and involved, nevertheless once one grasps its general structure it is easy to realise that the same procedure is repeated over and over to control the angular derivatives of all the various connection coefficients.

We go by induction. Let us suppose we have already obtained the estimates for $\mathcal{L}_O^{N-1}\mathcal{O}$, therefore we show how to control the \mathcal{L}_O^N derivatives of $\psi = \nabla\Omega^{-1}\text{tr}\chi + \Omega^{-1}\text{tr}\chi\eta$, (the details are in Section 9.)³¹ and discuss in the next subsection the general strategy.³²

Let us consider the transport equation of ψ , which can be derived immediately from the transport equation for $U = \Omega^{-1}\text{tr}\chi$,³³

$$\frac{\partial U}{\partial \nu} + \frac{\Omega \text{tr}\chi}{2}U + |\hat{\chi}|^2 = 0 \quad (3.4)$$

and the transport equation for η ,

$$\frac{\mathcal{D}}{\partial \nu}\eta + \Omega \frac{\text{tr}\chi}{2}\eta + \Omega \hat{\chi} \cdot \eta - \Omega \chi \cdot \underline{\eta} + \Omega \beta = 0, \quad (3.5)$$

$$\frac{\mathcal{D}}{\partial \nu}\psi + \frac{3}{2}\Omega \text{tr}\chi\psi = -\nabla|\hat{\chi}|^2 - \eta|\hat{\chi}|^2 - \Omega \hat{\chi} \cdot \psi + \text{tr}\chi(\hat{\chi} \cdot \underline{\eta}) - \text{tr}\chi\beta, \quad (3.6)$$

²⁹Recall as described carefully in (I) that an analytic solution of the system of equations (I;2.35), (I;2.36) is a solution of the Einstein equations provided the initial data satisfy equations (I;2.48), (I;2.49). It is important to recall, see also Theorem 4.5 of (I), that the solution $\Psi \in B_{\alpha, \rho''}$ and ρ'' is independent, due to the hyperbolicity, from the size of $\mathcal{K}(\Lambda_a, \Pi_a)$.

³⁰That this region can be foliated by a “double null cone foliation” follows as such a foliation is guaranteed even for much less regular solutions, see [Kl-Ni2] and [Ca-Ni2].

³¹ ψ is a combination of different connection coefficients. Its introduction is required to avoid some logarithmic divergences which will appear if we consider the transport equation for $\nabla \text{tr}\chi$, see details in [Kl-Ni2], Chapter 4. Once we control it with all its angular derivatives we control immediately $\nabla^N \text{tr}\chi$ with $N \geq 1$ and $\nabla^N \hat{\chi}$.

³²All the definitions not explicitly given here are in (I).

³³The trace part of the Raychaudhuri equation. The definition of U allow us to eliminate the dependance on $\omega = -2^{-1}\mathcal{P}_4 \log \Omega$ present in the equation for $\text{tr}\chi$.

which we rewrite as

$$\frac{\mathcal{D}}{\partial \nu} \psi + \frac{3}{2} \Omega \text{tr} \chi \psi + \Omega \hat{\chi} \cdot \psi + 2 \hat{\chi} \cdot (\nabla \hat{\chi}) = \mathcal{F}, \quad (3.7)$$

where

$$\mathcal{F} = -\text{tr} \chi \beta - \eta |\hat{\chi}|^2 + \text{tr} \chi \hat{\chi} \cdot \underline{\eta}. \quad (3.8)$$

From it we derive the structure equations for the angular derivatives³⁴ $\mathcal{L}_O^{N-1} \psi$ which will allow to control $\mathcal{L}_O^N \text{tr} \chi$ and $\mathcal{L}_O^{N-1} \nabla \hat{\chi}$.

$$\frac{\mathcal{D}}{\partial \nu} (\mathcal{L}_O^{N-1} \psi) + \frac{3}{2} \Omega \text{tr} \chi (\mathcal{L}_O^{N-1} \psi) = -\Omega \hat{\chi} \cdot (\mathcal{L}_O^{N-1} \psi) - \mathcal{L}_O^{N-1} \nabla |\hat{\chi}|^2 + \mathcal{L}_O^{N-1} \beta + \mathcal{F}_{l.o.} \quad (3.9)$$

where $\mathcal{F}_{l.o.}$ is a complicated quadratic expression,³⁵ see equation 13.8, of the connection coefficients and of the Riemann tensor components, but with no \mathcal{L}_O derivatives of order N .³⁶ The strategy is to use Gronwall inequality to prove the correct bound for the $|\mathcal{L}_O^{N-1} \psi|_{p,S}$ norm once we make inductive assumptions for all the lower derivatives of the connection coefficients to control $\mathcal{F}_{l.o.}$ and once we have estimates for the initial data connection coefficients. The more delicate part, to complete this procedure, is the control of $\mathcal{L}_O^{N-1} \nabla \hat{\chi}$ and of $\mathcal{L}_O^{N-1} \beta$. These two contributions have to be estimated separately.

To control $\mathcal{L}_O^{N-1} \nabla \hat{\chi}$, looking at the structure equations, see (I), Appendix to Section 2, we observe that, as we said in the introduction, there are not transport equations for $\hat{\chi}$ along the outgoing cones with no loss of derivatives, but there are the Hodge equations,³⁷

$$\text{div} \hat{\chi} = \frac{1}{2} \nabla \text{tr} \chi - \beta - \zeta \cdot \hat{\chi} + \frac{1}{2} \zeta \text{tr} \chi \equiv F^{(1)}$$

and from them one derives analogous Hodge systems for the angular derivatives $\mathcal{L}_O^{N-1} \nabla \hat{\chi}$. This will allow to get for $|\mathcal{L}_O^{N-1} \nabla \hat{\chi}|_{p,S}$ the estimate³⁸

$$|(\mathcal{L}_O^{N-1} \nabla \hat{\chi})|_{p,S} \leq c (|\Omega|_{\infty,S} |(\mathcal{L}_O^{N-1} \psi)|_{p,S} + |(\mathcal{L}_O^{N-1} \beta)|_{p,S}) + \mathcal{G}_{l.o.} \quad (3.10)$$

Remarks:

i) In order to obtain the equations 3.9, and 3.10 from the structure equations, we have to estimate the $|\cdot|_{p,S}$ norms of the commutators $[\mathcal{L}_O^K, \mathcal{D}_\nu]$, $[\mathcal{L}_O^K, \text{div}]$ and $[\mathcal{L}_O^K, \nabla]$ applied to ψ and $\nabla \hat{\chi}$; to do this we need estimates also for the first

³⁴See next subsection for its precise derivation.

³⁵ $l.o.$ stays for “lower order”.

³⁶Of order $N - 1$ for the Riemann components.

³⁷ ϵ is the totally antisymmetric tensor.

³⁸ $\mathcal{G}_{l.o.}$ denotes an estimate of terms with lower order derivatives ($< N$).

derivatives of the connection coefficients, which we include in the final version of Theorem 3.6.

ii) To be useful this estimate requires that we have an independent way of controlling the $|\cdot|_{p,S}$ norm of β . Once this is achieved the expression of 3.10 of $\mathcal{L}_O^{N-1}\nabla\hat{\chi}$ norm can be substituted in 3.9. After a careful examination of the structure of the terms in $\mathcal{F}_{l.o.}$, the right estimate for $|\mathcal{L}_O^{N-1}\psi|_{p,S}$ can be obtained and, finally, from the Hodge system the expected estimate for $|\hat{\mathcal{L}}_O^{N-1}\nabla\hat{\chi}|_{p,S}$ and, from the definition of ψ , the one of $|\mathcal{L}_O^{N-1}\nabla\text{tr}\chi|_{p,S}$.

Therefore, besides the transport equation and the Hodge system, we have to control $|\mathcal{L}_O^{N-1}\beta|_{p,S}$. The estimates for $\mathcal{L}_O^{N-1}\beta$ or, more in general, for the derivatives of the null Riemann components are based on the hyperbolic nature of the Einstein equations implying that we can obtain apriori estimates for “energy type” norms, as discussed at length,³⁹ in [Kl-Ni2]. This procedure reproduces itself for all the remaining connection coefficients, therefore we summarize in the next subsection the general strategy we use consisting, basically, of four different steps.⁴⁰

3.2 The general strategy for the angular derivatives

I) We have a transport equation for a quantity expressed in terms of the first derivatives of the connection coefficients⁴¹, let us denote it \mathcal{M} ; this implies an analogous equation for its \mathcal{L}_O^{N-1} derivatives which has, in general,⁴² the following expression:

$$\frac{\mathcal{D}(\mathcal{L}_O^{N-1}\mathcal{M})}{\partial\nu} + K\frac{\Omega\text{tr}\chi}{2}(\mathcal{L}_O^{N-1}\mathcal{M}) + \mathcal{O}\mathcal{L}_O^{N-1}\nabla\mathcal{O} + H(\mathcal{O})\mathcal{L}_O^{N-1}\Psi = \mathcal{K}_{l.o.} \quad (3.11)$$

where K is some fractional number, $H(\mathcal{O})\mathcal{L}_O^{N-1}\Psi$ denotes a linear expression in the connection coefficients times the \mathcal{L}_O^{N-1} derivatives of Riemann null components and $\mathcal{K}_{l.o.}$ collects all the terms which depend on lower order derivatives which can be estimated using the inductive assumptions.

II) The remaining term which depends on the highest order derivatives, $\mathcal{O}\mathcal{L}_O^{N-1}\nabla\mathcal{O}$, has the property that, in general, there is no transport equation for $\nabla\mathcal{O}$ without loss of derivatives, but \mathcal{O} satisfies an Hodge equation which, slightly symbolically, can be written as,

$$\text{div}\mathcal{O} = \mathcal{M} + \Psi + \mathcal{H}_{l.o.} \quad (3.12)$$

³⁹Up to the first derivatives of the Riemann tensor.

⁴⁰We describe it for the non underlined quantities as for the underlined ones the structure is exactly the same with the obvious modifications.

⁴¹or second derivatives...

⁴²See the details in Section 9.

where with Ψ we denote one or more Riemann null components and $\mathcal{H}_{l.o.}$ contains lower order terms. applying \mathcal{L}_O^{N-1} times Hodge system we obtain an equation, similar to 3.10, which allows an estimate $\mathcal{L}_O^{N-1}\nabla\mathcal{O}$ in terms of $\mathcal{L}_O^{N-1}\mathcal{M}$, provided we control the (norms of the) \mathcal{L}_O^{N-1} derivatives of the Riemann components Ψ .

III) The control of the \mathcal{L}_O derivatives of Ψ , up to $N-1$, is the third step required to get the result which, again, is present in all the connection coefficients norm estimates; it is implemented using the hyperbolic nature of the Einstein equations which shows itself in the apriori estimates for the energy-type L^2 norms, we denote them by Q norms, whose proof is in Section 4; these apriori estimates can be proved in the whole spacetime if the initial data are small, otherwise there will be a bounded region where they hold.⁴³

IV) Once we control the $N-1$ derivatives of Ψ we integrate 3.11 along the outgoing cones⁴⁴ and use the Gronwall's inequality to control $|\mathcal{L}_O^{N-1}\mathcal{M}|_{p,S}$ and finally again the Hodge system to control (the norm of) $\mathcal{L}_O^{N-1}\nabla\mathcal{O}$ and, subsequently, of $\mathcal{L}_O^N\mathcal{O}$.

V) As final last step we recover the estimates for $\nabla^{N-1}\mathcal{M}$ and $\nabla^N\mathcal{O}$ from the previous estimates

3.3 The last slice and the canonical foliation

A technical but important remark is appropriate here and will be discussed in more detail in the following subsection later on. To prove Theorem 3.1 it is crucial that we assume in the inductive hypothesis the correct asymptotic decay of the (derivatives of the) connection coefficients along the outgoing cones. To obtain it it turns out that the integration along the outgoing cones for all the not underlined quantities has to be done going backward,⁴⁵ starting from the greatest \underline{u} value, $\nu_* = \Pi_a$.

This implies we have first to know the estimates for all the connection coefficients on the last slice, $\underline{C}(\nu_*)$. This can be achieved using the incoming transport equations starting from C_0 at (λ_0, ν_*) and exploiting the energy estimates starting from the initial data surface \underline{C}_0 . The arising problem, proceeding in this way, is that some of these transport equations imply a loss of derivatives which would not allow an inductive procedure to control the higher derivatives. To overcome it one has to choose, in an appropriate way, on $\underline{C}(\nu_*) \equiv \underline{C}_*$ the shift function Ω (see (I) for its definition); this specifies on \underline{C}_* the function $u = u|_{\underline{C}_*}$ and a foliation on it. The outgoing cones in \mathcal{K} are then defined as the level hypersurfaces of a solution of the eikonal equations with $u|_{\underline{C}_*}$ as “initial data”. This defines the so called “double null cone canonical foliation”. We discuss carefully how to obtain this foliation in subsections 9.5, 9.6.1 and in subsection 14.11, but we also recall that this procedure is not new and is due to D.Christodoulou and

⁴³ Analogous to the time slab $R \times [0, T]$ of the Burger equation, see (I), Section 4.

⁴⁴ Or along the incoming cones depending on which transport equation for \mathcal{M} is without loss of derivatives.

⁴⁵ With the exception of ω and $\underline{\omega}$ whose role is inverted.

S. Klainerman who introduced it in [Ch-Kl] and was subsequently readapted in [Kl-Ni2] and [Ni] where more details are given.

Remark:

A contradiction seems to appear as we claim that to prove the analytic solution existence, we assign initial data on $C_0 \cup \underline{C}_0$ and, nevertheless, to prove the norm estimates of Lemma 3.1 we have to assign “final data” on the last slice $\underline{C}(\nu_)$. Nevertheless this contradiction is only apparent and the picture of the global strategy to prove the existence of a global analytic solution goes in the following way:*

We assign initial data on $C_0 \cup \underline{C}_0$ satisfying the appropriate norm estimates 3.31 and 3.32 and applying Cauchy-Kowalevski we prove, as discussed in (I), the existence of the solution in a small region; then we denote, see Theorem 3.1, $\mathcal{K}(\Lambda_a, \Pi_a) \equiv \{(\lambda, \nu) \in [\nu_0, \Lambda_a] \times [\lambda_0, \Pi_a]\}$ as the larger region where, with the assigned initial data, this analytic solution does exist. Observe that up to now the double null foliation of the existence region is not specified.

Introducing on the upper part $\underline{C}(\Pi_a)$ of the boundary of $\mathcal{K}(\Lambda_a, \Pi_a)$, the part we call in the previous subsection “the last slice” and denoted $\underline{C}(\nu_)$, an appropriate foliation defined through Ω_* , we prove, starting from $S(\lambda_0, \nu_*) \subset C_0$ and exploiting the energy norms norms from \underline{C}_0 , appropriate norm estimates for the not underlined connection coefficients on $\underline{C}(\nu_*)$ and using them and the transport equations on the outgoing cones we prove the norm estimates of Theorem 3.1; this is the crucial step to show that the region $\mathcal{K}(\Lambda_a, \Pi_a)$ can be extended implying, to avoid a contradiction, that this region is in fact unbounded.*

Of course this also implies that the double null foliation of this region required to prove its unboundedness is, see also [Kl-Ni2], the “double null canonical foliation”, that is the one determined by the foliation imposed on the last slice to obtain the correct “final data norms”.

Finally the last thing to observe is that, going back from the data on the last slice with the outgoing transport equations to \underline{C}_0 we obtain some norms bounds that the not underlined connection coefficients have to satisfy on \underline{C}_0 and we have to prove that these bounds are compatible with the previously assumed initial data; this is easy to prove for the following reasons: we can now assume all the transport equations along the outgoing cones without worrying anymore of loss of derivatives as we already have the estimates for all order derivatives and due to the fact that the final parameter ρ appearing in these estimates is smaller than the initial one, ρ_0 , it is immediate to realize that all these estimates are consistent with the initial data estimates.⁴⁶

3.4 The initial data conditions

To prove the inductive estimate for $|r^{N+2-\frac{2}{p}} \nabla^{N-1} \psi|_{p,S}$ and for the $|\cdot|_{p,S}$ norms of the remaining connection coefficients, Theorem 3.1, we need some assump-

⁴⁶In principle the foliation on \underline{C}_0 of the initial data is different from the foliation induced on \underline{C}_0 from the canonical foliation, but it can be proved, proceeding as in [Kl-Ni2] that, due to the fact that we are considering a “small initial data problem” the two induced foliations are near and the result follows.

tions on the initial data. This is a bit delicate as the initial data have to satisfy some constraint equations.⁴⁷ As it will be proved in detail later on, the strategy is first to define the initial data in a neighbourhood of $S_{(0)} \equiv C_0 \cap \underline{C}_0$ and then prove, again with a “Burger type” mechanism, that the analytic initial data can be defined on the whole $C_0 \cup \underline{C}_0$. The proof is a simplified version of the main proof, with the basic difference that in this case we, as we directly assign those connection coefficients which need the Hodge equations to be estimated, we don’t need the estimates of the null Riemann components, and hence of the energy type norms. For the same reason we do not need to integrate along C_0 “from above”. Moreover, the exponential factor, $\underline{\Gamma}(\lambda)$, in the initial data assumptions is different, namely we require,

$$\underline{\Gamma}_0(\lambda) = \tilde{C}_0 \frac{(\lambda - \lambda_0)}{\lambda \lambda_0} . \quad (3.13)$$

With suitable $\tilde{C}_0 < \hat{C}$.

Therefore we show in Section 7 that we can construct analytic initial data satisfying the following bounds for any J with $\delta_0 < \delta, \epsilon > 0$,

Theorem 3.2 *Assuming $\hat{\chi}$ on C_0 , $\underline{\hat{\chi}}$ and X on \underline{C}_0 , Ω on $C_0 \cup \underline{C}_0$ satisfying the following estimates,*

$$\begin{aligned} |r^{J+2+\epsilon-\frac{2}{p}} \nabla^J \log \Omega|_{p,S}(\lambda_0, \nu) &\leq C_3^{(0)} e^{(J-2)(\delta_0+\underline{\Gamma}_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,0,1}^J} \\ |r^{J+\frac{5}{2}+\epsilon-\frac{2}{p}} \nabla^J \hat{\chi}|_{p,S}(\lambda_0, \nu) &\leq C_1^{(0)} e^{(J-2)(\delta_0+\underline{\Gamma}_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,0,1}^J}, \\ ||\lambda|^{1+\epsilon} r^{1+J-\frac{2}{p}} \nabla^J \log \Omega|_{p,S}(\lambda, \nu_0) &\leq \underline{C}_3^{(0)} e^{(J-2)(\delta_0+\underline{\Gamma}_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,0,1}^J} \quad (3.14) \\ ||\lambda|^{\frac{3}{2}+\epsilon} r^{1+J-\frac{2}{p}} \nabla^J \underline{\hat{\chi}}|_{p,S}(\lambda, \nu_0) &\leq \underline{C}_1^{(0)} e^{(J-2)(\delta_0+\underline{\Gamma}_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,0,1}^J} \end{aligned}$$

Where we choose $C_{1,3}^{(0)}$ and $\underline{C}_{1,3}^{(0)}$ of order $O(\epsilon)$ for $J \leq 7$ and $O(1)$ otherwise, moreover we assume $\underline{\omega}$ have some definite expressions on S_0 , see [Ca-Ni1] and 7.7, finally, that on S_0 suitable conditions are satisfied for the angular derivatives of the $\{\gamma, \zeta, tr\chi, tr\underline{\chi}, \underline{\omega}, \mathbf{K}\}$ see equations 7.4, then the following relations hold:
On C_0 :

$$\begin{aligned} |r^{J+2-\frac{2}{p}} \nabla^J U|_{p,S} &\leq C_0^{(0)} e^{(J-2)(\delta_0+\underline{\Gamma}_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_0^J} \\ |r^{J+\frac{5}{2}+\epsilon-\frac{2}{p}} \nabla^J \hat{\chi}|_{p,S} &\leq C_1^{(0)} e^{(J-2)(\delta_0+\underline{\Gamma}_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_0^J} \\ |r^{J+2+\epsilon-\frac{2}{p}} \nabla^J \omega|_{p,S} &\leq C_2^{(0)} e^{(J-2)(\delta_0+\underline{\Gamma}_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_0^J} \end{aligned}$$

⁴⁷See (I) for a detailed discussion of the constraint equations.

$$\begin{aligned}
|r^{J+2-\frac{2}{p}} \nabla^J \zeta|_{p,S} &\leq C_4^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_0^J} \\
|r^{J+2-\frac{2}{p}} \nabla^J \underline{U}|_{p,S} &\leq C_5^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_0^J} \\
||\lambda_0| r^{1+J-\frac{2}{p}} \nabla^J \underline{\hat{\chi}}|_{p,S} &\leq C_6^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_0^J} \\
||\lambda_0| r^{1+J-\frac{2}{p}} \nabla^J \underline{\omega}|_{p,S} &\leq C_7^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_0^J}
\end{aligned} \tag{3.15}$$

On \underline{C}_0 :

$$\begin{aligned}
|r^{J+1-\frac{2}{p}} |\lambda| \nabla^J \underline{U}|_{p,S} &\leq \underline{C}_0^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_0^J} \\
|r^{J+1-\frac{2}{p}} |\lambda|^{\frac{3}{2}+\epsilon} \nabla^J \underline{\hat{\chi}}|_{p,S} &\leq \underline{C}_1^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_0^J} \\
|r^{J+1-\frac{2}{p}} |\lambda|^{1+\epsilon} \nabla^J \underline{\omega}|_{p,S} &\leq \underline{C}_2^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_0^J} \\
|r^{J+2-\frac{2}{p}} \nabla^J \underline{U}|_{p,S} &\leq \underline{C}_4^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_0^J} \\
|r^{J+2-\frac{2}{p}} \nabla^J \underline{\hat{\chi}}|_{p,S} &\leq \underline{C}_5^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_0^J} \\
|r^{J+2-\frac{2}{p}} \nabla^J \underline{\omega}|_{p,S} &\leq \underline{C}_6^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_0^J} \\
|r^{J+2-\frac{2}{p}} \nabla^J \zeta|_{p,S} &\leq \underline{C}_7^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_0^J}
\end{aligned} \tag{3.16}$$

with the constants $C_i^{(0)}$, $\underline{C}_i^{(0)}$ of order $O(\varepsilon)$ for $J \leq 7$ and $O(1)$ otherwise.

Proof: The proof is substantially the content of [Ca-Nil], we retrace and improve that result in Section 7.

Remarks:

i) Notice that, as said before, for the initial data, we do not need the energy estimates, as on the initial data we already possess all the tangential derivatives of χ , $\underline{\chi}$, Ω and hence for ω and $\underline{\omega}$ at any order and so we do not have to use the Hodge estimates and hence the estimates for all the tangential derivatives of the null Riemann components. This is a great advantage, as we do not have to retrace all the machinery we have to use in the internal mixed derivatives estimates.

ii) Notice moreover that in order to prove Theorem 3.2 we do not need the full tangential derivatives of the free connection coefficients but only the angular ones in the hypothesis 3.14., we require the full ones in order to estimate also the mixed derivatives, see Theorem 3.4

iii) We have to require a slightly better decay for some of the connection coefficients, adding an $\epsilon > 0$ on the decay for r , in order to assure the boundedness of the \mathcal{Q} norms.

iv) With respect to the construction of the initial data provided in [Ca-Ni1], we provide the ζ on S_0 instead that on the upper part of the incoming cone \underline{C}_0 , this is an improvement.

From these estimates we obtain the estimates for the \mathcal{L}_O derivatives on the initial data, which will be needed to obtain the energy estimates of the null Riemann components. Hence we have to prove the following estimates:

Theorem 3.3 *Under the assumptions of Theorem 3.2, the following estimates hold*

On C_0 :

$$\begin{aligned}
|r^{2-\frac{2}{p}} \mathcal{L}_O^J U|_{p,S} &\leq C_{0,0}^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{\frac{5}{2}+\epsilon-\frac{2}{p}} \mathcal{L}_O^J \hat{\chi}|_{p,S} &\leq C_{0,1}^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2+\epsilon-\frac{2}{p}} \mathcal{L}_O^J \omega|_{p,S} &\leq C_{0,2}^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}} \mathcal{L}_O^J \zeta|_{p,S} &\leq C_{0,3}^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}} \mathcal{L}_O^J \text{tr} \underline{\chi}|_{p,S} &\leq C_{0,4}^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
||\lambda_0| r^{1-\frac{2}{p}} \mathcal{L}_O^J \hat{\underline{\chi}}|_{p,S} &\leq C_{0,5}^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
||\lambda_0| r^{1-\frac{2}{p}} \mathcal{L}_O^J \underline{\omega}|_{p,S} &\leq C_{0,6}^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J}
\end{aligned} \tag{3.17}$$

On \underline{C}_0 :

$$\begin{aligned}
|r^{1-\frac{2}{p}} |\lambda| \mathcal{L}_O^J U|_{p,S} &\leq \underline{C}_{0,0}^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{1-\frac{2}{p}} |\lambda|^{\frac{3}{2}+\epsilon} \mathcal{L}_O^J \hat{\underline{\chi}}|_{p,S} &\leq \underline{C}_{0,1}^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{1-\frac{2}{p}} |\lambda|^{1+\epsilon} \mathcal{L}_O^J \underline{\omega}|_{p,S} &\leq \underline{C}_{0,2}^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}} \mathcal{L}_O^J U|_{p,S} &\leq \underline{C}_{0,3}^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}} \mathcal{L}_O^J \hat{\underline{\chi}}|_{p,S} &\leq \underline{C}_{0,4}^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}} \mathcal{L}_O^J \omega|_{p,S} &\leq \underline{C}_{0,5}^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}} \mathcal{L}_O^J \zeta|_{p,S} &\leq \underline{C}_{0,6}^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J}.
\end{aligned} \tag{3.18}$$

With $\rho_{0,1} < \rho_0$, $C_i^{(0)} < C_{0,i}^{(0)}$, $\underline{C}_i^{(0)} < \underline{C}_{0,i}^{(0)}$, $C_{0,i}^{(0)} \leq C_{0,i+1}^{(0)}$ and $\underline{C}_{0,i}^{(0)} \leq \underline{C}_{0,i+1}^{(0)}$ and all the constants of order $O(\varepsilon)$ for $J \leq 7$ and $O(1)$ otherwise. Moreover we have similar estimates for the first derivatives of the connection coefficients:

On C_0 :

$$\begin{aligned}
|r^{3-\frac{2}{p}} \mathcal{L}_O^{J-1} \nabla U|_{p,S} &\leq C_{1,0}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{\frac{7}{2} + \epsilon - \frac{2}{p}} \mathcal{L}_O^{J-1} \nabla \hat{\chi}|_{p,S} &\leq C_{1,1}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{3+\epsilon-\frac{2}{p}} \mathcal{L}_O^{J-1} \nabla \omega|_{p,S} &\leq C_{1,2}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{3-\frac{2}{p}} \mathcal{L}_O^{J-1} \nabla \zeta|_{p,S} &\leq C_{1,3}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{3-\frac{2}{p}} \mathcal{L}_O^{J-1} \nabla \text{tr} \underline{\chi}|_{p,S} &\leq C_{1,4}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
||\lambda_0| r^{2-\frac{2}{p}} \mathcal{L}_O^{J-1} \nabla \hat{\chi}|_{p,S} &\leq C_{1,5}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
||\lambda_0| r^{2-\frac{2}{p}} \mathcal{L}_O^{J-1} \nabla \omega|_{p,S} &\leq C_{1,6}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J}
\end{aligned} \tag{3.19}$$

On \underline{C}_0 :

$$\begin{aligned}
|r^{2-\frac{2}{p}} |\lambda| \mathcal{L}_O^{J-1} \nabla \underline{U}|_{p,S} &\leq \underline{C}_{1,0}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}} |\lambda|^{\frac{3}{2} + \epsilon} \mathcal{L}_O^{J-1} \nabla \hat{\chi}|_{p,S} &\leq \underline{C}_{1,1}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}} |\lambda|^{1+\epsilon} \mathcal{L}_O^{J-1} \nabla \omega|_{p,S} &\leq \underline{C}_{1,2}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{3-\frac{2}{p}} \mathcal{L}_O^{J-1} \nabla U|_{p,S} &\leq \underline{C}_{1,3}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{3-\frac{2}{p}} \mathcal{L}_O^{J-1} \nabla \hat{\chi}|_{p,S} &\leq \underline{C}_{1,4}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{3-\frac{2}{p}} \mathcal{L}_O^{J-1} \nabla \omega|_{p,S} &\leq \underline{C}_{1,5}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{3-\frac{2}{p}} \mathcal{L}_O^{J-1} \nabla \zeta|_{p,S} &\leq \underline{C}_{1,6}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J}.
\end{aligned} \tag{3.20}$$

With $C_{1,i}^{(0)} \leq C_{1,i+1}^{(0)}$ and $\underline{C}_{1,i}^{(0)} \leq \underline{C}_{1,i+1}^{(0)}$ and all the constant of order $O(\varepsilon)$ for $J \leq 7$ and $O(1)$ otherwise.

Proof: The proof is a repetition of lemmas 4.1 and 4.2 applied now to the connection coefficients.

Remark:

Notice that the hierarchy among the constants C is obtained from the sequence we adopt to prove the inequalities 3.17, 3.18, 3.19 and 3.20.

In order to prove Theorem 1.2 we need also the estimates for the other tangential derivatives of the initial data null cones, namely $\mathfrak{D}_\nu^J \nabla^p \mathcal{O}$ for C_0 and $\mathfrak{D}_\lambda^J \nabla^p \mathcal{O}$ for \underline{C}_0 . This is the result of the following theorem:

Theorem 3.4 *Assuming the hypothesis of Theorem 3.2, the following estimates hold for the $\mathfrak{D}_\nu^p \nabla^q$ derivatives of the connection coefficients,*

$$\begin{aligned}
|r^{1+p+q+\sigma(P)-\frac{2}{p}} \mathfrak{D}_\nu^p \nabla^q U|_{p,S} &\leq F_1 \left(\frac{(p+q)!}{(p+q)^\alpha} \frac{e^{(p+q-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^{p+q}} \right) \\
|r^{1+p+q+\sigma(P)-\frac{2}{p}} \mathfrak{D}_\nu^p \nabla^q \text{tr} \chi|_{p,S} &\leq F_2 \left(\frac{(p+q)!}{(p+q)^\alpha} \frac{e^{(p+q-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^{p+q}} \right) \\
|r^{2+p+q-\frac{2}{p}} \mathfrak{D}_\nu^J \nabla^p \hat{\chi}|_{p,S} &\leq F_3 \left(\frac{(J+P)!}{(p+q)^\alpha} \frac{e^{(p+q-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^{p+q}} \right) \\
|r^{2+p+q-\frac{2}{p}} \mathfrak{D}_\nu^p \nabla^q \eta|_{p,S} &\leq F_4 \left(\frac{(p+q)!}{(p+q)^\alpha} \frac{e^{(p+q-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^{p+q}} \right) \\
|r^{2+p+q-\frac{2}{p}} \mathfrak{D}_\nu^p \nabla^q \zeta|_{p,S} &\leq F_5 \left(\frac{(p+q+1)!}{(p+q+1)^\alpha} \frac{e^{(p+q-1)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^{p+q+1}} \right) \\
|r^{2+p+q-\frac{2}{p}} \mathfrak{D}_\nu^J \nabla^P U|_{p,S} &\leq \underline{F}_1 \left(\frac{(p+q)!}{(p+q)^\alpha} \frac{e^{(p+q-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^{p+q}} \right) \\
||\lambda| r^{1+p+q-\frac{2}{p}} \mathfrak{D}_\nu^J \nabla^P \hat{\chi}|_{p,S} &\leq \underline{F}_3 \left(\frac{(p+q)!}{(p+q)^\alpha} \frac{e^{(p+q-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^{p+q}} \right) \\
|r^{2+p+q-\frac{2}{p}} \mathfrak{D}_\nu^J \nabla^P \omega|_{p,S} &\leq F_6 \left(\frac{(p+q+1)!}{(p+q+1)^\alpha} \frac{e^{(p+q-1)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^{p+q+1}} \right) \\
|r^{2+p+q-\frac{2}{p}} \mathfrak{D}_\nu^J \nabla^P \underline{\omega}|_{p,S} &\leq \underline{F}_6 \left(\frac{(p+q)!}{(p+q)^\alpha} \frac{e^{(p+q-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^{p+q}} \right).
\end{aligned} \tag{3.21}$$

and the analogous estimates with \mathfrak{D}_λ instead of \mathfrak{D}_ν on \underline{C}_0 .

Proof: The proof is in section 7.

Remark:

As already noticed in Remark i) of Theorem 3.2 for the angular derivatives, here we do not need the energy estimates, as on the initial data we already possess all the mixed derivatives of χ on C_0 , $\underline{\chi}$ and X on \underline{C}_0 , of Ω and hence for ω and $\underline{\omega}$ on $C_0 \cup \underline{C}_0$ at any order and so we do not have to use the Hodge estimates and hence the estimates for all the mixed derivatives of the null Riemann components. Notice that in order to prove Theorem 3.4 we need the full tangential derivatives of the free connection coefficients assumed in the hypothesis of Theorem 3.2.

Clearly we can also state the analogous of inequalities 3.17, 3.18, 3.19 and 3.20 for the mixed derivatives, in this case, in order to estimate the right energy

norms we have to estimate the $\mathcal{L}_S^p \mathcal{L}_O^q \mathcal{O}$, the Lie derivatives of the connection coefficients.⁴⁸:

Theorem 3.5 *Under the hypotesis of Theorem 3.4, the following inequalities hold, with $p + q = J$*

On C_0 :

$$\begin{aligned}
|r^{2-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q U|_{p,S} &\leq F_{0,0}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{\frac{5}{2}+\epsilon-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \hat{\chi}|_{p,S} &\leq F_{0,1}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2+\epsilon-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \omega|_{p,S} &\leq F_{0,2}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \zeta|_{p,S} &\leq F_{0,3}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \text{tr} \chi|_{p,S} &\leq F_{0,4}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
||\lambda_0| r^{1-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \hat{\chi}|_{p,S} &\leq F_{0,5}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
||\lambda_0| r^{1-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \omega|_{p,S} &\leq F_{0,6}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J}
\end{aligned} \tag{3.22}$$

On \underline{C}_0 :

$$\begin{aligned}
|r^{1-\frac{2}{p}} |\lambda| \mathcal{L}_S^p \mathcal{L}_O^q U|_{p,S} &\leq \underline{F}_{0,0}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{1-\frac{2}{p}} |\lambda|^{\frac{3}{2}+\epsilon} \mathcal{L}_S^p \mathcal{L}_O^q \hat{\chi}|_{p,S} &\leq \underline{F}_{0,1}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{1-\frac{2}{p}} |\lambda|^{1+\epsilon} \mathcal{L}_S^p \mathcal{L}_O^q \omega|_{p,S} &\leq \underline{F}_{0,2}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q U|_{p,S} &\leq \underline{F}_{0,3}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \hat{\chi}|_{p,S} &\leq \underline{F}_{0,4}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \omega|_{p,S} &\leq \underline{F}_{0,5}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \zeta|_{p,S} &\leq \underline{F}_{0,6}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J}.
\end{aligned} \tag{3.23}$$

With $\rho_{0,1} < \rho_0$, $F_{0,i}^{(0)} \leq F_{0,i+1}^{(0)}$ and $\underline{F}_{0,i}^{(0)} \leq \underline{F}_{0,i+1}^{(0)}$, all the constant $O(\varepsilon)$ for $J \leq 7$ and $O(1)$ otherwise.

⁴⁸We can assume the constant C and \underline{C} be the same of the inequalities for the angular derivatives

Moreover we have similar estimates for the first derivatives of the connection coefficients, with $p + q = J - 1$:

On C_0 :

$$\begin{aligned}
|r^{3-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \nabla U|_{p,S} &\leq F_{1,0}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{\frac{7}{2}+\epsilon-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \nabla \hat{\chi}|_{p,S} &\leq F_{1,1}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{3+\epsilon-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \nabla \omega|_{p,S} &\leq F_{1,2}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{3-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \nabla \zeta|_{p,S} &\leq F_{1,3}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{3-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \nabla \text{tr} \chi|_{p,S} &\leq F_{1,4}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
||\lambda_0| r^{2-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \nabla \hat{\chi}|_{p,S} &\leq F_{1,5}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
||\lambda_0| r^{2-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \nabla \underline{\omega}|_{p,S} &\leq F_{1,6}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J}
\end{aligned} \tag{3.24}$$

On \underline{C}_0 :

$$\begin{aligned}
|r^{2-\frac{2}{p}} |\lambda| \mathcal{L}_S^p \mathcal{L}_O^q \nabla U|_{p,S} &\leq \underline{F}_{1,0}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}} |\lambda|^{\frac{3}{2}+\epsilon} \mathcal{L}_S^p \mathcal{L}_O^q \nabla \hat{\chi}|_{p,S} &\leq \underline{F}_{1,1}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}} |\lambda|^{1+\epsilon} \mathcal{L}_S^p \mathcal{L}_O^q \nabla \omega|_{p,S} &\leq \underline{F}_{1,2}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{3-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \nabla U|_{p,S} &\leq \underline{F}_{1,3}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{3-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \nabla \hat{\chi}|_{p,S} &\leq \underline{F}_{1,4}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{3-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \nabla \omega|_{p,S} &\leq \underline{F}_{1,5}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{3-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \nabla \zeta|_{p,S} &\leq \underline{F}_{1,6}^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J}.
\end{aligned} \tag{3.25}$$

With $F_{1,i}^{(0)} \leq \underline{F}_{1,i}^{(0)}$ and $\underline{F}_{1,i}^{(0)} \leq F_{1,i+1}^{(0)}$ and all the constant of order $O(\varepsilon)$ for $J \leq 7$ and $O(1)$ otherwise.

Proof: The proof of inequalities 3.24 and 3.25 follows as in the proof of lemmas 4.1 and 4.2, with the obvious modifications.

3.5 The covariant angular derivatives of the connection coefficients, the precise version of Theorem 3.1

We write here the precise estimates for the covariant angular derivatives of the connection coefficients, the detailed proof of this result is in Section 9.

First we state a modified version of the main theorem, where the ∇ derivatives are substituted with the \mathcal{L}_O derivatives, in order to exploit the energy estimates for the null Riemann components.

Theorem 3.6 *Under the hypothesis stated in Theorem 3.1 and, more precisely, the assumptions on the connection coefficients, 3.15, 3.16, on $C_0 \cup \underline{C}_0$ given in the previous subsection 3.4, the following estimates hold on \mathcal{K} ,*

$$\begin{aligned}
|r^{3-\frac{2}{p}} \mathcal{L}_O^{J-1} \nabla U|_{p,S} &\leq C_0 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{3-\frac{2}{p}} \mathcal{L}_O^{J-1} \nabla \hat{\chi}|_{p,S} &\leq C_1 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{3-\frac{2}{p}} \mathcal{L}_O^{J-1} \nabla \omega|_{p,S} &\leq C_2 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{3-\frac{2}{p}} \mathcal{L}_O^{J-1} \nabla \zeta|_{p,S} &\leq C_3 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{3-\frac{2}{p}} \mathcal{L}_O^{J-1} \nabla \underline{U}|_{p,S} &\leq \underline{C}_0 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}} |\lambda| \mathcal{L}_O^{J-1} \nabla \hat{\chi}|_{p,S} &\leq \underline{C}_1 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}} |\lambda| \mathcal{L}_O^{J-1} \nabla \underline{\omega}|_{p,S} &\leq \underline{C}_2 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} .
\end{aligned} \tag{3.26}$$

Moreover, the following estimates for the connection coefficients hold:

$$\begin{aligned}
|r^{2-\frac{2}{p}} \mathcal{L}_O^J U|_{p,S} &\leq C_0 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}} \mathcal{L}_O^J \hat{\chi}|_{p,S} &\leq C_1 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}} \mathcal{L}_O^J \omega|_{p,S} &\leq \underline{C}_2 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}} \mathcal{L}_O^J \underline{U}|_{p,S} &\leq \underline{C}_0 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{1-\frac{2}{p}} |\lambda| \mathcal{L}_O^J \hat{\chi}|_{p,S} &\leq \underline{C}_1 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{1-\frac{2}{p}} |\lambda| \mathcal{L}_O^J \underline{\omega}|_{p,S} &\leq C_2 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} .
\end{aligned}$$

With a suitable choice of the constants C_i of order $O(\varepsilon)$ for $J \leq 7$ and $O(1)$ otherwise and of the radius of convergence $\rho_{0,1} < \rho_0$, $\rho_{0,1}$ depending only the first s derivatives of the initial data, with $s \leq 7$.

Proof: The proof in Section 9

Finally, the following estimates hold for the $|\cdot|_{\infty,S}$ norms of the connection coefficients,

Corollary 3.7 *Under the hypothesis stated in Theorem 3.1, the Following estimates hold on \mathcal{K} :*

$$\begin{aligned}
|r^2 \mathcal{L}_O^{J-1} U|_{\infty,S} &\leq C_0 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^2 \mathcal{L}_O^{J-1} \hat{\chi}|_{\infty,S} &\leq C_1 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^2 \mathcal{L}_O^{J-1} \omega|_{\infty,S} &\leq \underline{C}_2 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^2 \mathcal{L}_O^{J-1} \zeta|_{\infty,S} &\leq \underline{C}_3 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^2 \mathcal{L}_O^{J-1} \underline{U}|_{\infty,S} &\leq \underline{C}_0 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r|\lambda| \mathcal{L}_O^{J-1} \hat{\chi}|_{\infty,S} &\leq \underline{C}_1 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r|\lambda| \mathcal{L}_O^{J-1} \underline{\omega}|_{\infty,S} &\leq C_2 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} .
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
|r^3 \mathcal{L}_O^{J-2} \nabla U|_{\infty,S} &\leq C_0 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^3 \mathcal{L}_O^{J-2} \nabla \hat{\chi}|_{\infty,S} &\leq C_1 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^3 \mathcal{L}_O^{J-2} \nabla \omega|_{\infty,S} &\leq \underline{C}_2 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^3 \mathcal{L}_O^{J-2} \nabla \zeta|_{\infty,S} &\leq \underline{C}_3 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^3 \mathcal{L}_O^{J-2} \nabla \underline{U}|_{\infty,S} &\leq \underline{C}_0 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^2 |\lambda| \mathcal{L}_O^{J-2} \nabla \hat{\chi}|_{\infty,S} &\leq \underline{C}_1 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^2 |\lambda| \mathcal{L}_O^{J-2} \nabla \underline{\omega}|_{\infty,S} &\leq C_2 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} .
\end{aligned} \tag{3.28}$$

Proof: The proof is in section 9

Remarks:

i) We will prove the estimates for $\eta = \zeta + \nabla \log \Omega$. The estimates for ζ follow in a straightforward way.

ii) The estimates 3.26, 3.27 and 3.28, have not the same constants C_i and \underline{C}_i , we call them in the same way only for the sake of simplicity.

Now we can state the final form of Theorem 3.1.

Theorem 3.8 *Under the hypothesis stated in Theorem 3.1 and, more precisely the assumptions, on the connection coefficients, 3.15, 3.16, on $C_0 \cup \underline{C}_0$ given in the previous subsection 3.4, the following estimates hold on \mathcal{K} ⁴⁹*

$$\begin{aligned}
|r^{J+2-\frac{2}{p}} \nabla^J U|_{p,S} &\leq C_0 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho^J} \\
|r^{J+2-\frac{2}{p}} \nabla^J \hat{\chi}|_{p,S} &\leq C_1 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho^J} \\
|r^{J+2-\frac{2}{p}} \nabla^J \omega|_{p,S} &\leq \underline{C}_2 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho^J} \\
|r^{J+2-\frac{2}{p}} \nabla^J \zeta|_{p,S} &\leq \underline{C}_3 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho^J} \\
|r^{J+2-\frac{2}{p}} \nabla^J \underline{U}|_{p,S} &\leq \underline{C}_0 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho^J} \\
|r^{J+1-\frac{2}{p}} |\lambda| \nabla^J \hat{\chi}|_{p,S} &\leq \underline{C}_1 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho^J} \\
|r^{J+1-\frac{2}{p}} |\lambda| \nabla^J \underline{\omega}|_{p,S} &\leq C_2 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho^J} .
\end{aligned} \tag{3.29}$$

with $\rho < \rho_{0,1}$, ρ depending only the first s derivatives of the initial data, with $s \leq 7$ and with a suitable choice of the constants C_i , \underline{C}_i of order $O(\varepsilon)$ for $J \leq 7$ and $O(1)$ otherwise

Proof: The proof follows from Theorem 3.6 and is equivalent, for the connection coefficients, to lemma 10.4 of section 10.1.5 for the null Riemann components .

3.6 \mathbb{D}_ν and \mathbb{D}_λ derivatives and the mixed ones for the connection coefficients

To complete our proof we need to estimate the higher derivatives with respect to e_3 and e_4 of the connection coefficients and consequently the higher derivatives with respect to e_3 and e_4 of the Riemann components. One could try to use the Weyl equations to get \mathbb{D}_3^J and \mathbb{D}_4^J derivatives from the angular ones, for instance, considering the β component, exploiting the equation

$$\mathbb{D}_3 \beta + \text{tr} \chi \beta = \nabla \rho + [2\underline{\omega} \beta + {}^* \nabla \sigma + 2\hat{\chi} \cdot \underline{\beta} + 3(\eta \rho + {}^* \eta \sigma)] \tag{3.30}$$

⁴⁹ All the remaining quantities are defined in Theorem 3.1.

We get $\mathfrak{D}_3\beta$ term from the angular derivatives, nevertheless iterating this mechanism for the β component, in the expression of $\mathfrak{D}_3^K\beta$, with $K > 4$, a term $\nabla^3\mathfrak{D}_3^{K-3}\underline{\alpha}$ is present and to control it⁵⁰ we have to extend the family of the Q norms we are considering. More precisely we have to extend the Q norms also to terms with $\tilde{W} = \hat{\mathcal{L}}_S^J W$ with $J \leq N-1$ with $S = \underline{u}e_4 + ue_3$, beside those with $\hat{\mathcal{L}}_O^J W$. More in general to control all the mixed (tangential) derivatives with \mathfrak{D}_4 , \mathfrak{D}_3 q times and the angular ones, ∇ , p times with $p+q \leq N-1$, we have to define the \mathcal{Q} norms analogous to those defined in [Kl-Ni2], see also definitions 15.3 and 15.4, for all the Weyl tensors $\tilde{W}_{p,q} = \mathcal{L}_S^q \hat{\mathcal{L}}_O^p W$ with $p+q \leq N-1$. In order to estimate these new norms we have to reproduce all the machinery adopted to estimate angular derivatives of both, the connection coefficients and the null Riemann components.

Finally we will prove the following theorem, analogous to theorem 3.6.

Theorem 3.9 *Under the estimates on the initial data of Theorem 3.4, the following estimates hold on \mathcal{K} ,⁵¹ with $p+q = J$ ⁵²:*

$$\begin{aligned}
|r^{2-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q U|_{p,S} &\leq C_{0,0} e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{\frac{5}{2}+\epsilon-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \hat{\chi}|_{p,S} &\leq C_{0,1} e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2+\epsilon-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \omega|_{p,S} &\leq C_{0,2} e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \zeta|_{p,S} &\leq C_{0,3} e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \text{tr}\chi|_{p,S} &\leq C_{0,4} e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
||\lambda_0| r^{1-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \hat{\chi}|_{p,S} &\leq C_{0,5} e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
||\lambda_0| r^{1-\frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^q \underline{\omega}|_{p,S} &\leq C_{0,6} e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
\\
|r^{1-\frac{2}{p}} |\lambda| \mathcal{L}_S^p \mathcal{L}_O^q \underline{U}|_{p,S} &\leq \underline{C}_{0,0} e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} \\
|r^{1-\frac{2}{p}} |\lambda|^{\frac{3}{2}} \mathcal{L}_S^p \mathcal{L}_O^q \hat{\chi}|_{p,S} &\leq \underline{C}_{0,1} e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J}
\end{aligned} \tag{3.31}$$

⁵⁰Remember that we do not have a transport equation for $\underline{\alpha}$ along the incoming cones.

⁵¹all the remaining quantities are defined in Theorem 3.1.

⁵²We can assume the constant C and \underline{C} be the same of the inequalities for the angular derivatives

$$\begin{aligned}
|r^{1-\frac{2}{p}}|\lambda|^1\mathcal{L}_S^p\mathcal{L}_O^q\omega|_{p,S} &\leq \underline{C}_{0,2}e^{(J-2)\delta}e^{J\Gamma(\lambda)}\frac{J!}{J^\alpha}\frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}}\mathcal{L}_S^p\mathcal{L}_O^qU|_{p,S} &\leq \underline{C}_{0,3}e^{(J-2)\delta}e^{(J-2)\Gamma(\lambda)}\frac{J!}{J^\alpha}\frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}}\mathcal{L}_S^p\mathcal{L}_O^q\hat{\chi}|_{p,S} &\leq \underline{C}_{0,4}e^{(J-2)\delta}e^{(J-2)\Gamma(\lambda)}\frac{J!}{J^\alpha}\frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}}\mathcal{L}_S^p\mathcal{L}_O^q\omega|_{p,S} &\leq \underline{C}_{0,5}e^{(J-2)\delta}e^{(J-2)\Gamma(\lambda)}\frac{J!}{J^\alpha}\frac{1}{\rho_{0,1}^J} \\
|r^{2-\frac{2}{p}}\mathcal{L}_S^p\mathcal{L}_O^q\zeta|_{p,S} &\leq \underline{C}_{0,6}e^{(J-2)\delta}e^{(J-2)\Gamma(\lambda)}\frac{J!}{J^\alpha}\frac{1}{\rho_{0,1}^J}.
\end{aligned} \tag{3.32}$$

With $\rho_{0,1}$, depending only the first s derivatives of the initial data, with $s \leq 7$ and with a suitable choice of the constants $C_{0,i}$ of order $O(\varepsilon)$ for $J \leq 7$ and $O(1)$ otherwise.

Proof: The proof goes in the same way as done for the $\mathcal{L}_O^J\mathcal{O}$ estimates for the connection coefficients, It suffices to notice that, with U a generic connection coefficient.

$$(\mathcal{L}_S U)(e_A) = \partial_S(U(e_A)) - U([S, e_A]) = D_S U(e_A) + U(D_{e_A} S) \tag{3.33}$$

Therefore

$$\mathcal{L}_S U(\cdot) = D_S U(\cdot) + \mathbf{g}(e_B, D.S)U(e_B) = D_S U(\cdot) + (U \cdot \mathcal{M})(\cdot) \tag{3.34}$$

where

$$\mathcal{M}(\cdot, \cdot) = \mathbf{g}(\cdot, D.S) = u\chi(\cdot, \cdot) + \underline{u}\chi(\cdot, \cdot) \quad . \tag{3.35}$$

and we recall that

$$\mathcal{L}_O V = D_O V + \mathcal{H} \cdot V, \tag{3.36}$$

where

$$\mathcal{H}(\cdot, \cdot) = \mathbf{g}(\cdot, \nabla \cdot O) \quad . \tag{3.37}$$

Hence we can pass from \mathcal{L}_O to ∇_O and from \mathcal{L}_S to \mathfrak{D}_S in a formally identical way as discussed in details in Sections 9 and in its appendix, see Section 14, but now for the $\mathcal{L}_S \mathcal{L}_O$ derivatives, exploiting the smallness of the initial data 3.22 and 3.23 and the smallness of the extended \mathcal{Q} norms, see section 4.3.

Remark:

It is important to observe that the mixed derivatives estimates can also be obtained in a different way, without solving again the the transport equations for $\mathcal{L}_S^p \mathcal{L}_O^q \mathcal{O}$. in fact as we already have all the estimates for the $\mathcal{L}_O^p \mathcal{O}$ terms, for any p , we can write

$$\mathcal{L}_S^p \mathcal{L}_O^p \mathcal{O} = \mathcal{L}_S^{q-1} \mathcal{L}_O^p \mathcal{L}_S \mathcal{O} + \mathcal{L}_O^{q-1} [\mathcal{L}_S^p \mathcal{L}_O^q] \mathcal{O}$$

and write explicitly $\mathcal{L}_S \mathcal{O}$ using the structure equations. In this way all the terms on the right side can be estimate, with a long but standard computations, without solving explicitly the transport equations. This is possible as now we can use all the structure equation without worrying of the lost of derivatives, as we already control all the angular derivatives estimates. This procedure is analogous to what has been done for the mixed derivatives the initial data, see 3.4, and points out the fact that the central role played by the estimates of the angular derivatives.

The next step is to estimates of the $\mathbb{D}_\nu^p \nabla^q$ mixed derivatives of the connection coefficients.

Lemma 3.1 *Let us assume the estimates of theorem 3.9, then the connection coefficients \mathcal{O} satisfy the following estimates for $p + q = J$ on \mathcal{K} , with $\rho_1 < \rho < \rho_{0,1}$ depending only the first s derivatives of the initial data, with $s \leq 7$ and with a suitable choice of the D_4 constant of order $O(\varepsilon)$ for $J \leq 7$ and $O(1)$ otherwise.*

$$||\lambda|_{\underline{\phi}(\Psi)} r^{\phi(\mathcal{O})+q-\frac{2}{p}} D_S^p \nabla^k \mathcal{O}|_{L^p(S)} \leq D_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_1^J}, \quad (3.38)$$

with $\phi(\mathcal{O})$ the suitable weights related to the different connection coefficients, see 3.22, 3.23. ⁵³

Proof: The proof is analogous to the one of lemma 4.3 but now for the connection coefficients.

We have now to pass from \mathbb{D}_S to \mathbb{D}_ν and \mathbb{D}_λ this is the content of the following lemma, analogous to lemma 4.4

Lemma 3.2 *Given the assumptions of Lemma 3.1 the following estimates hold on \mathcal{K} ,*

$$\begin{aligned} ||\lambda|_{\underline{\phi}(\mathcal{O})} r^{\phi(\mathcal{O})+(J-1)-\frac{2}{p}} D_\nu^p \nabla^q \mathcal{O}|_{L^p(S)} &\leq D_5 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_2^p \rho_1^k} \\ ||\lambda|_{\underline{\phi}(\Psi)+p} r^{\phi(\mathcal{O})+k-\frac{2}{p}} D_\lambda^p \nabla^q \mathcal{O}|_{L^p(S)} &\leq D_5 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_2^p \rho_1^k}. \end{aligned} \quad (3.39)$$

With $\rho_2 \leq \rho_1$ depending only the first s derivatives of the initial data, with $s \leq 7$ and with a suitable choice of the constants D_i of order $O(\varepsilon)$ for $J \leq 7$ and $O(1)$ otherwise.

Proof: The proof is a repetition of Lemma 4.4, but now for the connection coefficients .

Remark:

The reason of having $\rho_2 < \rho$ in the previous estimates of lemma 3.4 and 3.5 is due to the fact that in proving these estimates we need the estimates for the mixed derivatives of the Riemann components. These estimates are obtained

⁵³For $J = 0$ ρ becomes $\rho - \underline{\rho}$, see [Kl-Ni2].

from the \mathcal{Q} norm estimates associated to the $\hat{\mathcal{L}}_S^k \hat{\mathcal{L}}_O^p W$ Weyl fields. The problem arises when from $\hat{\mathcal{L}}_S$ or D_S we move to D_ν or to D_λ as this requires lowering ρ_1 to ρ_2 . It is important to remark that we are allowed to do this as we do not have to use the mixed derivatives of the connection coefficients in a bootstrap mechanism. In fact the mixed derivative appear in the error estimates as mixed derivatives of π, \tilde{q} or more in general of the connection coefficients, but in this case they appear with D_S or $\hat{\mathcal{L}}_S$.

3.7 General mixed derivatives

Let us discuss the situation when the mixed derivatives are disposed in an arbitrary way, we start considering a specific case as the structure of the proof will be the same for all the remaining connection coefficients. Therefore we show how to bound the norm of the following term,

$$\nabla^{p_0} \mathfrak{D}_\nu \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} U$$

where

$$U = \Omega^{-1} \text{tr} \chi \quad , \quad \sum_{s=0}^J p_s = P \quad , \quad J + P = N \quad .$$

Lemma 3.3 *Given the assumptions of Lemma 3.1 the following estimates hold on \mathcal{K}_ρ , with $\rho_2 < \rho$ depending only the first s derivatives of the initial data, with $s \leq 7$ and with a suitable choice of the constants \hat{F} of order $O(\varepsilon)$ for $J \leq 7$ and $O(1)$ otherwise.*

$$|r^{1+J+P+\sigma(P)-\frac{2}{p}} \nabla^{p_0} \mathfrak{D}_\nu \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \text{tr} \chi|_{p,S} \leq \hat{F} \left(\frac{(J+P)!}{(J+P)^\alpha} \frac{e^{(J+P-2)(\delta+\Gamma(\lambda))}}{\rho_2^{J+P}} \right) . \quad (3.40)$$

Proof: *The proof is in appendix 16.*

The proof, can be extended without crucial modification to all the remaining connection coefficients. Notice that to prove the general mixed derivatives estimates for the connection coefficients we need to control also the general mixed derivatives estimates for the Riemann components but at a lower order of derivatives, these estimates can be achieved directly by the knowledge of the mixed derivatives of the connection coefficients already at our disposal by induction, see lemma 3.6. We can state hence the following lemma:

Lemma 3.4 *Under the assumptions of Lemma 3.1 the following estimates hold on \mathcal{K}_ρ , with $\rho_2 < \rho$ depending only the first s derivatives of the initial data, with $s \leq 7$ and with a suitable choice of the constants \hat{F}_i of order $O(\varepsilon)$ for $J \leq 7$ and $O(1)$ otherwise.*

$$\begin{aligned}
|r^{1+J+P+\sigma(P)-\frac{2}{p}} \nabla^{p_0} \mathfrak{D}_\nu \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \text{tr} \chi|_{p,S} &\leq \hat{F}_1 \left(\frac{(J+P)!}{(J+P)^\alpha} \frac{e^{(J+P-2)(\delta+\underline{\Gamma}(\lambda))}}{\rho_2^{J+P}} \right) \\
|r^{2+J+P-\frac{2}{p}} \nabla^{p_0} \mathfrak{D}_\nu \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \hat{\chi}|_{p,S} &\leq \hat{F}_2 \left(\frac{(J+P)!}{(J+P)^\alpha} \frac{e^{(J+P-2)(\delta+\underline{\Gamma}(\lambda))}}{\rho_2^{J+P}} \right) \\
|r^{2+J+P-\frac{2}{p}} \nabla^{p_0} \mathfrak{D}_\nu \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \zeta|_{p,S} &\leq \hat{F}_3 \left(\frac{(J+P+1)!}{(J+P+1)^\alpha} \frac{e^{(J+P-1)(\delta+\underline{\Gamma}(\lambda))}}{\rho_2^{J+P+1}} \right) \\
|r^{2+J+P-\frac{2}{p}} \nabla^{p_0} \mathfrak{D}_\nu \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \omega|_{p,S} &\leq \hat{F}_4 \left(\frac{(J+P+1)!}{(J+P+1)^\alpha} \frac{e^{(J+P-1)(\delta+\underline{\Gamma}(\lambda))}}{\rho_2^{J+P+1}} \right) \\
|r^{1+J+P+\sigma(P)-\frac{2}{p}} \nabla^{p_0} \mathfrak{D}_\nu \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \text{tr} \underline{\chi}|_{p,S} &\leq \hat{F}_5 \left(\frac{(J+P)!}{(J+P)^\alpha} \frac{e^{(J+P-2)(\delta+\underline{\Gamma}(\lambda))}}{\rho_2^{J+P}} \right) \\
||\lambda| r^{1+J+P-\frac{2}{p}} \nabla^{p_0} \mathfrak{D}_\nu \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \hat{\chi}|_{p,S} &\leq \hat{F}_6 \left(\frac{(J+P)!}{(J+P)^\alpha} \frac{e^{(J+P-2)(\delta+\underline{\Gamma}(\lambda))}}{\rho_2^{J+P}} \right) \\
||\lambda| r^{1+J+P-\frac{2}{p}} \nabla^{p_0} \mathfrak{D}_\nu \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \underline{\omega}|_{p,S} &\leq \hat{F}_7 \left(\frac{(J+P)!}{(J+P)^\alpha} \frac{e^{(J+P-2)(\delta+\underline{\Gamma}(\lambda))}}{\rho_2^{J+P}} \right) \quad . \quad (3.41)
\end{aligned}$$

Proceeding exactly in the same way with the obvious modifications we can prove analogous estimates for the “tangential” derivatives for the incoming cones. In this case one has to recall that in the weight factors each \mathfrak{D}_λ derivative brings a $|\lambda|$ factor instead of a r factor. A specific care has to be done to control the mixed derivatives of $\hat{\omega}$. The final result is summarized in the following lemma:

Lemma 3.5 *Under the assumptions of Lemma 3.1 the following estimates hold on \mathcal{K} , with $\rho_2 < \rho$ depending only the first s derivatives of the initial data, with $s \leq 7$ and with a suitable choice of the constants \hat{F}_i of order $O(\varepsilon)$ for $J \leq 7$ and $O(1)$ otherwise.*

$$\begin{aligned}
||\lambda|^J r^{1+P+\sigma(P)-\frac{2}{p}} \nabla^{p_0} \mathfrak{D}_\lambda \nabla^{p_1} \mathfrak{D}_\lambda \nabla^{p_2} \dots \mathfrak{D}_\lambda \nabla^{p_{J-1}} \mathfrak{D}_\lambda \nabla^{p_J} \text{tr} \chi|_{p,S} &\leq \hat{E}_1 \left(\frac{(J+P)!}{(J+P)^\alpha} \frac{e^{(J+P-2)(\delta+\underline{\Gamma}(\lambda))}}{\rho_2^{J+P}} \right) \\
||\lambda|^J r^{2+P-\frac{2}{p}} \nabla^{p_0} \mathfrak{D}_\lambda \nabla^{p_1} \mathfrak{D}_\lambda \nabla^{p_2} \dots \mathfrak{D}_\lambda \nabla^{p_{J-1}} \mathfrak{D}_\lambda \nabla^{p_J} \hat{\chi}|_{p,S} &\leq \hat{E}_2 \left(\frac{(J+P)!}{(J+P)^\alpha} \frac{e^{(J+P-2)(\delta+\underline{\Gamma}(\lambda))}}{\rho_2^{J+P}} \right) \\
||\lambda|^J r^{2+P-\frac{2}{p}} \nabla^{p_0} \mathfrak{D}_\lambda \nabla^{p_1} \mathfrak{D}_\lambda \nabla^{p_2} \dots \mathfrak{D}_\lambda \nabla^{p_{J-1}} \mathfrak{D}_\lambda \nabla^{p_J} \zeta|_{p,S} &\leq \hat{E}_3 \left(\frac{(J+P+1)!}{(J+P+1)^\alpha} \frac{e^{(J+P-1)(\delta+\underline{\Gamma}(\lambda))}}{\rho_2^{J+P+1}} \right) \\
||\lambda|^J r^{2+P-\frac{2}{p}} \nabla^{p_0} \mathfrak{D}_\lambda \nabla^{p_1} \mathfrak{D}_\lambda \nabla^{p_2} \dots \mathfrak{D}_\lambda \nabla^{p_{J-1}} \mathfrak{D}_\lambda \nabla^{p_J} \omega|_{p,S} &\leq \hat{E}_4 \left(\frac{(J+P)!}{(J+P)^\alpha} \frac{e^{(J+P-2)(\delta+\underline{\Gamma}(\lambda))}}{\rho_2^{J+P}} \right) \\
||\lambda|^J r^{1+P+\sigma(P)-\frac{2}{p}} \nabla^{p_0} \mathfrak{D}_\lambda \nabla^{p_1} \mathfrak{D}_\lambda \nabla^{p_2} \dots \mathfrak{D}_\lambda \nabla^{p_{J-1}} \mathfrak{D}_\lambda \nabla^{p_J} \text{tr} \underline{\chi}|_{p,S} &\leq \hat{E}_5 \left(\frac{(J+P)!}{(J+P)^\alpha} \frac{e^{(J+P-2)(\delta+\underline{\Gamma}(\lambda))}}{\rho_2^{J+P}} \right) \\
||\lambda|^{J+1} r^{1+P-\frac{2}{p}} \nabla^{p_0} \mathfrak{D}_\lambda \nabla^{p_1} \mathfrak{D}_\lambda \nabla^{p_2} \dots \mathfrak{D}_\lambda \nabla^{p_{J-1}} \mathfrak{D}_\lambda \nabla^{p_J} \hat{\chi}|_{p,S} &\leq \hat{E}_6 \left(\frac{(J+P)!}{(J+P)^\alpha} \frac{e^{(J+P-2)(\delta+\underline{\Gamma}(\lambda))}}{\rho_2^{J+P}} \right)
\end{aligned}$$

$$||\lambda|^{J+1} r^{1+P-\frac{2}{p}} \nabla^{p_0} \mathfrak{D}_\lambda \nabla^{p_1} \mathfrak{D}_\lambda \nabla^{p_2} \dots \mathfrak{D}_\lambda \nabla^{p_{J-1}} \mathfrak{D}_\lambda \nabla^{p_J} \underline{\omega}|_{p,S} \leq \hat{F}_7 \left(\frac{(J+P+1)!}{(J+P+1)^\alpha} \frac{e^{(J+P-1)(\delta+\Gamma(\lambda))}}{\rho_2^{J+P+1}} \right). \quad (3.42)$$

In the next lemma we discuss and prove the estimates for the general mixed derivatives of the Riemann null components required in the proof of the previous Lemmas 3.4 and 3.5.

Lemma 3.6 *Under the assumptions of Lemma 3.1 the following estimates hold on \mathcal{K} , with $\rho_2 < \rho$ depending only the first s derivatives of the initial data, with $s \leq 7$ and with a suitable choice of the constants \hat{F}_i of order $O(\varepsilon)$ for $J \leq 7$ and $O(1)$ otherwise.*

$$||\lambda|^J r^{1+P+\sigma(P)-\frac{2}{p}} \nabla^{p_0} \mathfrak{D}_\lambda \nabla^{p_1} \mathfrak{D}_\lambda \nabla^{p_2} \dots \mathfrak{D}_\lambda \nabla^{p_{J-1}} \mathfrak{D}_\lambda \nabla^{p_J} \text{tr}\chi|_{p,S} \leq \hat{F}_1 \left(\frac{(J+P)!}{(J+P)^\alpha} \frac{e^{(J+P-2)(\delta+\Gamma(\lambda))}}{\rho_2^{J+P}} \right) \quad (3.43)$$

Proof: The proof is in appendix 16. Collecting the results we obtain the precise version of theorem 1.2

Theorem 3.10 *Under the hypothesis of theorem 3.2, where the initial data $\|\Phi^{(0)}\|_s$, $s \leq 7$ are chosen such that the \mathcal{Q}^0 norms on $C_0 \cup \underline{C}_0$ are sufficiently small, then the solution is analytic and in $\mathcal{B}_{\alpha,\rho_2}$ in all the domain of dependence of $C_0 \cup \underline{C}_0$.*

4 The \mathcal{Q} “energy norms” and their boundedness

As discussed in the introduction and in subsection 3.2, step III, to control the angular derivatives of the connection coefficients we need to control the norms of the Riemann components, Ψ ; this is achieved using the hyperbolicity of the Einstein equations which implies the existence of a priori energy estimates. This is exploited through the introduction of the \mathcal{Q} norms defined in the following subsections, a generalization of the analogous norms defined in [Ch-Kl] and [Kl-Ni2]. Repeating the arguments discussed there and also doing some extra work, we prove that these \mathcal{Q} norms satisfy some appropriate bounds depending on the initial data’s ones. The central point is that now we have to control an infinite family of energy-type norms, we denote $\mathcal{Q}^{(J-2)}(\lambda, \nu)$ and $\underline{\mathcal{Q}}^{(J-2)}(\lambda, \nu)$, for all J ,⁵⁴ as we need to control all order derivatives to prove the analyticity of the solution, but, nevertheless, the region where all these a priori estimates are valid depends only on the region where the \mathcal{Q} norms with $J = 2$ are bounded. This, as discussed in remarks below Theorem 1.2 and in subsection 3.4, depends on the smallness of the initial data derivatives with $J = 2$ ⁵⁵; if these are of order $O(\varepsilon)$, with suitable ε , we can prove that these a priori estimates, and therefore the a priori estimates for all the J , hold everywhere, a crucial step to prove the global existence of our analytic solution.

Remarks:

- i) It is exactly to control these \mathcal{Q} norms everywhere that the smallness of the initial data is crucial. In fact the proof of Theorem 3.1 is valid, in the region where the \mathcal{Q} norms are bounded, independently from the initial data smallness.
- ii) Comparing the strategy discussed in this paper with the analogous result proved in the toy model presented in (I), “the Burger case”, one could ask where the analogous of the t dependance for the Burger inductive assumptions, appears in the present case. Recall that in that case the assumptions we prove are, denoting with u a solution of the Burger equation,

$$\|D^J u(t)\|_{L^2} \leq C_0 e^{(J-2)\gamma t} \frac{J!}{J^\alpha} \frac{1}{\rho^J} . \quad (4.1)$$

The presence of the factor $e^{(J-2)\gamma t}$ is an indication of the fact that we are considering a local solution and therefore that these norms increase in time as we do not have an appropriate decay of the solution to make the local solution global.⁵⁶

In the present case the situation is different as in the Einstein equation case we can prove global a priori estimates, provided the initial data are “small”. In fact we use the “generalized” \mathcal{Q} norms where the Bel-Robinson tensor is saturated by the K vector fields which guarantees the appropriate decays; nevertheless in

⁵⁴ $J - 2$ is enough control the J derivatives norms of the connection coefficients as discussed in detail later on.

⁵⁵ We recall that, in order to bound the \mathcal{Q}^0 norms we need to control the tangential derivatives of the connection coefficients up to order $J \leq 7$ see [Ca-Nil]

⁵⁶ Recall that in the Burger case the estimate 4.1 does not hold for any t as it can be proved that some norms will blow up in a finite time.

the local case it is enough to use the Q norms “saturated” with the T vector fields. In this case when we try to obtain the control of the Q norms we have an inequality, formally, of the following type

$$Q(C(\lambda)) = Q(C_0) + \text{Error} = Q(C_0) + \int_0^\lambda d\lambda' |F(\mathcal{O})| Q(C(\lambda'))$$

where with $F(\mathcal{O})$ we indicate terms, depending on the connection coefficients, which we can estimate through the inductive assumptions. Therefore, assuming we control the sup norm of \mathcal{O} , the Gronwall inequality produces an exponential in time which, to be not harmful, requires that a decaying factor in r be present in the inductive assumption. An analogous decay factor should be present when we estimate those Q norms made along $\underline{C}(\nu)$, decay factor which is also required to close the estimates. This remark should explain that, if we do not restrict to small initial data, in our inductive assumption an exponential factor, $e^{(J-2)\gamma(|\lambda|)}$ should be present. Again, if the initial data are sufficiently small and, as it follows, the decay of the connection coefficients is sufficiently good, the previous exponential factor turns out to be substituted by the bounded exponential factor, see 3.3, $e^{(J-2)(\Gamma(\lambda))}$.

Observe also that we cannot use the Bianchi equations for the Riemann null components to get direct information on their tangential derivatives due to the fact that in these equations there is a loss of tangential derivatives.⁵⁷ This is an important point as this is exactly the loss of derivatives avoided using the Q norms. This is exactly where, in the present case, the hyperbolicity plays the role. In fact the boundedness of energy allows to control $\mathcal{L}_O^{N-1}\Psi$ on a generic $S(\lambda, \nu)$ in terms of the analogous quantities on the initial hypersurface.⁵⁸

iii) As done before, we proceed looking for the global existence proof; from the previous discussions it should be clear that relaxing the smallness assumptions for the first derivatives initial data, all the subsequent machinery will allow to prove the analyticity in the bounded region \mathcal{K} where the a priori energy type estimates hold.

In the following subsection we define the Q norms we need to control the angular derivatives of the null Riemann components and show how to prove their estimates.

4.1 The Q “energy norms” required to control the angular derivatives .

The Q norms we write here and which we will prove bounded are those needed to control all orders of the Lie angular derivatives of the Riemann components,

⁵⁷Moreover we do not even have transport equations for α on the outgoing cones and for $\underline{\alpha}$ on the incoming ones.

⁵⁸The absence of the loss of (tangential) derivatives is due to the fact that in the “error estimate” for the Q we use $D^\mu W_{\mu\nu\rho\sigma} = 0$ which corresponds to the use of $\square u = F$ in the wave equation.

at their turn required to control, as discussed before, the angular derivatives of the connection coefficients. Subsequently, as discussed in section 3.6 when we estimate the remaining “tangential” derivatives of the connection coefficients we need that some other \mathcal{Q} norms,

We define, for $J \geq 2$,⁵⁹

$$\begin{aligned}\mathcal{Q}^{(J-2)}(\lambda, \nu) &= \mathcal{Q}_1^{(J-2)}(\lambda, \nu) + \mathcal{Q}_2^{(J-2)}(\lambda, \nu) \\ \underline{\mathcal{Q}}^{(J-2)}(\lambda, \nu) &= \underline{\mathcal{Q}}_1^{(J-2)}(\lambda, \nu) + \underline{\mathcal{Q}}_2^{(J-2)}(\lambda, \nu)\end{aligned}\quad (4.2)$$

where,

$$\begin{aligned}\mathcal{Q}_1^{(J-2)}(\lambda, \nu) &\equiv \sum_{i_1, i_2, \dots, i_{J-2}}^{\{1,3\}} \int_{C(\lambda) \cap V(\lambda, \nu)} Q(\hat{\mathcal{L}}_T(\hat{\mathcal{L}}_{O_{(i_1)}} \hat{\mathcal{L}}_{O_{(i_2)}} \cdots \hat{\mathcal{L}}_{O_{(i_{J-2})}} W))(\bar{K}, \bar{K}, \bar{K}, e_4) \\ &+ \sum_{i_0, i_1, i_2, \dots, i_{J-2}}^{\{1,3\}} \int_{C(\lambda) \cap V(\lambda, \nu)} Q(\hat{\mathcal{L}}_{O_{(i_0)}}(\hat{\mathcal{L}}_{O_{(i_1)}} \hat{\mathcal{L}}_{O_{(i_2)}} \cdots \hat{\mathcal{L}}_{O_{(i_{J-2})}} W))(\bar{K}, \bar{K}, T, e_4) .\end{aligned}$$

we recall that $Q(\hat{\mathcal{L}}_O W) = \sum_{i=1}^3 Q(\hat{\mathcal{L}}_{O_{(i)}} W)$. Therefore with our definitions

$$\begin{aligned}\mathcal{Q}_1^{(J-2)}(\lambda, \nu) &\equiv \int_{C(\lambda) \cap V(\lambda, \nu)} Q(\hat{\mathcal{L}}_T(\hat{\mathcal{L}}_O^{J-2} W))(\bar{K}, \bar{K}, \bar{K}, e_4) \\ &+ \int_{C(\lambda) \cap V(\lambda, \nu)} Q(\hat{\mathcal{L}}_O(\hat{\mathcal{L}}_O^{J-2} W))(\bar{K}, \bar{K}, T, e_4) \\ \mathcal{Q}_2^{(J-2)}(\lambda, \nu) &\equiv \int_{C(\lambda) \cap V(\lambda, \nu)} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T(\hat{\mathcal{L}}_O^{J-2} W))(\bar{K}, \bar{K}, \bar{K}, e_4) \\ &+ \int_{C(\lambda) \cap V(\lambda, \nu)} Q(\hat{\mathcal{L}}_O^2(\hat{\mathcal{L}}_O^{J-2} W))(\bar{K}, \bar{K}, T, e_4) \\ &+ \int_{C(\lambda) \cap V(\lambda, \nu)} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T(\hat{\mathcal{L}}_O^{J-2} W))(\bar{K}, \bar{K}, \bar{K}, e_4)\end{aligned}\quad (4.3)$$

$$\begin{aligned}\underline{\mathcal{Q}}_1^{(J-2)}(\lambda, \nu) &\equiv \sup_{V(\lambda, \nu) \cap C_0} |r^3 \bar{\rho}(\hat{\mathcal{L}}_O^{J-2} W)|^2 + \int_{\underline{C}(\nu) \cap V(\lambda, \nu)} Q(\hat{\mathcal{L}}_T(\hat{\mathcal{L}}_O^{J-2} W))(\bar{K}, \bar{K}, \bar{K}, e_3) \\ &+ \int_{\underline{C}(\nu) \cap V(\lambda, \nu)} Q(\hat{\mathcal{L}}_O(\hat{\mathcal{L}}_O^{J-2} W))(\bar{K}, \bar{K}, T, e_3) \\ \underline{\mathcal{Q}}_2^{(J-2)}(\lambda, \nu) &\equiv \int_{\underline{C}(\nu) \cap V(\lambda, \nu)} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T(\hat{\mathcal{L}}_O^{J-2} W))(\bar{K}, \bar{K}, \bar{K}, e_3) \\ &+ \int_{\underline{C}(\nu) \cap V(\lambda, \nu)} Q(\hat{\mathcal{L}}_O^2(\hat{\mathcal{L}}_O^{J-2} W))(\bar{K}, \bar{K}, T, e_3) \\ &+ \int_{\underline{C}(\nu) \cap V(\lambda, \nu)} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T(\hat{\mathcal{L}}_O^{J-2} W))(\bar{K}, \bar{K}, \bar{K}, e_3)\end{aligned}\quad (4.4)$$

⁵⁹For the definitions of the vector fields O, K, T and the modified Lie derivatives $\hat{\mathcal{L}}$. see [Kl-Ni2] Chapter 3.

where $V(\lambda, \nu) = J^{(+)}(S(\lambda_0, \nu_0) \cap J^{(-)}(S(\lambda, \nu)))$. Analogous expressions hold for the initial data norms defined on $C_0 \cup \underline{C}_0 = C(\lambda_0) \cup \underline{C}(\nu_0, [\lambda_0, \lambda_*])$ which we denote

$$\begin{aligned}\mathcal{Q}_{(0)}^{(J-2)}(\lambda_0, \nu) &= \mathcal{Q}_{(0),1}^{(J-2)}(\lambda_0, \nu) + \mathcal{Q}_{(0),2}^{(J-2)}(\lambda_0, \nu) \\ \underline{\mathcal{Q}}_{(0)}^{(J-2)}(\lambda, \nu_0) &= \underline{\mathcal{Q}}_{(0),1}^{(J-2)}(\lambda, \nu_0) + \underline{\mathcal{Q}}_{(0),2}^{(J-2)}(\lambda, \nu_0) .\end{aligned}\quad (4.5)$$

Assuming the initial data sufficiently small, in the sense previously discussed of the “smallness” of the derivatives up to $J \leq 7^{60}$ we have to prove, mimicking [Kl-Ni2] that,

$$\mathcal{H}^{(N-2)}(\lambda, \nu) \equiv \sum_{J=2}^N \left(\mathcal{Q}^{(J-2)}(C(\lambda; [\nu_0, \nu])) + \underline{\mathcal{Q}}^{(J-2)}(\underline{C}(\nu; [\lambda_0, \lambda])) \right) \quad (4.6)$$

satisfies, everywhere the following bound,

$$\mathcal{H}^{(N-2)}(\lambda, \nu)^{\frac{1}{2}} \leq C^{*(1)} \left(\frac{(N)!}{(N)^\alpha} \frac{e^{(N-2)(\delta + \Gamma(\lambda))}}{\rho_{0,1}^N} \right) , \quad (4.7)$$

where defining $C^{(0)}$ a constant which bounds all the initial data constants, see 3.31 and 3.32, $C^{(1)}$ does the same for the constants in the inductive assumptions. The detailed statement of this result is in subsection 10.1.3, Theorem 10.1, and its proof in the Appendix to Section 10.

4.2 The estimates of the covariant angular derivatives of the null Riemann components

We are now in good position to control the angular derivatives of the Riemann tensor, whose null components are $\Psi \equiv \{\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}\}$,⁶¹ in the whole region \mathcal{K} ; this is the content of the following theorem,

Theorem 4.1 *Let us assume that the initial data connection coefficients satisfy, with all their angular derivatives, the estimates 3.15, 3.16 ; let us assume that the connection coefficients satisfy, in the internal region, \mathcal{K} , with all their angular derivatives up to order $J-1$ the estimates 3.29, then the following estimates hold for the $J-1$ angular derivatives of Riemann tensor null components,*
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$$\left| |\lambda|^{\phi(\Psi)} r^{\phi(\Psi) + (J-1) - \frac{2}{p}} \mathcal{L}_O^{J-1} \Psi \right|_{p,S} \leq C^{(1)} e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} , \quad (4.8)$$

⁶⁰Otherwise assuming $|\Pi|$ and $|\Lambda|$ sufficiently small.

⁶¹See (I) and [Kl-Ni2] for the complete definitions.

⁶²Again, if the first derivatives of the initial data are order $O(\varepsilon)$, then the estimates with $J \leq 7$ have the same structure of 4.8, but with the constant $C^{(1)} = O(\varepsilon_0)$ and $\varepsilon_0^2 < \varepsilon < \varepsilon_0$.

with $C^{(1)}$ of order $O(\varepsilon)$ for $J \leq 7$, of order $O(1)$ otherwise, depending on ν_0 and smaller than the constant related to the estimates of the connection coefficients, and

$$\begin{aligned} \phi(\alpha) &= \phi(\beta) = \frac{7}{2}, \quad \phi(\rho) = \phi(\sigma) = 3, \quad \phi(\underline{\beta}) = 2, \quad \phi(\underline{\alpha}) = 1 \\ \underline{\phi}(\alpha) &= \underline{\phi}(\beta) = 0, \quad \underline{\phi}(\rho) = \underline{\phi}(\sigma) = \frac{1}{2}, \quad \underline{\phi}(\underline{\beta}) = \frac{3}{2}, \quad \underline{\phi}(\underline{\alpha}) = \frac{5}{2}. \end{aligned} \quad (4.9)$$

Proof: This long proof is divided in various steps which we list here and discuss in any detail in sections 10 and 15.

Step 1: We assume that the initial data connection coefficients satisfy the estimates 3.15, 3.16.

By these assumptions we prove the following estimates for the various null components of the Riemann tensor:

$$\begin{aligned} ||\lambda|_{\underline{\phi}(\Psi)}^{\phi(\Psi)} r^{\phi(\Psi)+\epsilon+(J-\frac{2}{p})} \nabla^{J-1} \Psi(R)|_{p,S}(\lambda_0, \nu) &\leq \hat{C}_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)}}{\rho_0^J}, \quad (4.10) \\ ||\lambda|_{\underline{\phi}(\Psi)}^{\phi(\Psi)} r^{\phi(\Psi)+\epsilon+(J-\frac{2}{p})} \nabla^{J-1} \Psi(R)|_{p,S}(\lambda, \nu_0) &\leq \hat{C}_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)}}{\rho_0^J}. \end{aligned}$$

With $\hat{C}_{0,0}$ of order $O(\varepsilon)$ for $J \leq 7$, of order $O(1)$ otherwise. From these estimates we derive an analogous estimates for the derivatives with respect to the O rotation generator fields, namely

$$\begin{aligned} ||\lambda|_{\underline{\phi}(\Psi)}^{\phi(\Psi)} r^{\phi(\Psi)+\epsilon+(J-\frac{2}{p})} \nabla_O^{J-1} \Psi(R)|_{p,S}(\lambda_0, \nu) &\leq |O|_\infty^{J-1} \hat{C}_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)}}{\rho_{0,1}^J} \\ ||\lambda|_{\underline{\phi}(\Psi)}^{\phi(\Psi)} r^{\phi(\Psi)+\epsilon+(J-\frac{2}{p})} \nabla_O^{J-1} \Psi(R)|_{p,S}(\lambda, \nu_0) &\leq |O|_\infty^{J-1} \hat{C}_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)}}{\rho_{0,1}^J}, \end{aligned} \quad (4.11)$$

Namely we prove the following lemma:

Lemma 4.1 *Under the estimates 3.15, 3.16 for the connection coefficients and 4.10 for the null Riemann components on $C_0 \cup \underline{C}_0$, the following estimates for the ∇_O derivatives of the null Riemann components hold,*

$$||\lambda|_{\underline{\phi}(\Psi)}^{\phi(\Psi)} r^{\phi(\Psi)+\epsilon+(J-1)-\frac{2}{p}} \nabla_O^{J-1} \Psi(R)|_{p,S} \Big|_{C_0 \cup \underline{C}_0} \leq |O|_\infty^{J-1} \hat{C}_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)}}{\rho_{0,1}^J}, \quad (4.12)$$

where $\rho_{0,1} < \rho_0$ and the $\epsilon > 0$ in the weight factor is required to have the \mathcal{Q} norms of the initial data finite, see remark below Theorem 1.2.

Proof: See appendix to section 10⁶³.

⁶³Remember that, as $O = \sum_{i=1}^3 O^{(i)}$ we have to consider also a factor 3^{J-1} in the $|O|_\infty^{J-1}$ norms. We do not write explicitly it to avoid cumbersome notations.

Subsequently we derive an estimate for the $\mathcal{L}_O^{J-1}\Psi(R)$ derivatives, an estimate for the $\Psi(\mathcal{L}_O^{J-1}R)$ and finally an estimate for $\Psi(\hat{\mathcal{L}}_O^{J-1}R)$, in this case we can exploit the initial data estimates for the \mathcal{L}_O derivatives of the connection coefficients, see 3.17, 3.18, 3.19 and 3.20, and avoid the changes of the constant $c_{0,0}$ and of the radius of convergence $\rho_{0,1}$, we list here,

$$\begin{aligned} ||\lambda|_{\underline{\mathcal{C}}}^{\phi(\Psi)} r^{\phi(\Psi)+\epsilon-\frac{2}{p}} \mathcal{L}_O^{J-1}\Psi(R)|_{p,S}(\lambda_0, \nu) &\leq \hat{C}_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)}}{\rho_{0,1}^J} \quad (4.13) \\ ||\lambda|_{\underline{\mathcal{C}}}^{\phi(\Psi)} r^{\phi(\Psi)+\epsilon-\frac{2}{p}} \mathcal{L}_O^{J-1}\Psi(R)|_{p,S}(\lambda, \nu_0) &\leq \hat{C}_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)}}{\rho_{0,1}^J}, \end{aligned}$$

$$\begin{aligned} ||\lambda|_{\underline{\mathcal{C}}}^{\phi(\Psi)} r^{\phi(\Psi)+\epsilon-\frac{2}{p}} \Psi(\mathcal{L}_O^{J-1}R)|_{p,S}(\lambda_0, \nu) &\leq \hat{C}_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)}}{\rho_{0,1}^J} \quad (4.14) \\ ||\lambda|_{\underline{\mathcal{C}}}^{\phi(\Psi)} r^{\phi(\Psi)+\epsilon-\frac{2}{p}} \Psi(\mathcal{L}_O^{J-1}R)|_{p,S}(\lambda, \nu_0) &\leq \hat{C}_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)(\nu)}}{\rho_{0,1}^J}, \end{aligned}$$

and finally

$$\begin{aligned} ||\lambda|_{\underline{\mathcal{C}}}^{\phi(\Psi)} r^{\phi(\Psi)+\epsilon-\frac{2}{p}} \Psi(\hat{\mathcal{L}}_O^{J-1}R)|_{p,S}(\lambda_0, \nu) &\leq \hat{C}_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)}}{\rho_{0,1}^J} \quad (4.15) \\ ||\lambda|_{\underline{\mathcal{C}}}^{\phi(\Psi)} r^{\phi(\Psi)+\epsilon-\frac{2}{p}} \Psi(\hat{\mathcal{L}}_O^{J-1}R)|_{p,S}(\lambda, \nu_0) &\leq \hat{C}_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)}}{\rho_{0,1}^J}, \end{aligned}$$

These results are summarised in lemma

Lemma 4.2 *Under the assumptions of lemma 4.1 on the (initial data) connection coefficients and the previous estimates for $\nabla_O^{J-1}\Psi(R)$, the following bounds hold,*

$$||\lambda|_{\underline{\mathcal{C}}}^{\phi(\Psi)} r^{\phi(\Psi)+\epsilon-\frac{2}{p}} \mathcal{L}_O^{J-1}\Psi(R)|_{p,S} \Big|_{C_0 \cup \underline{\mathcal{C}}_0} \leq \hat{C}_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)}}{\rho_{0,1}^J} \quad (4.16)$$

moreover

$$||\lambda|_{\underline{\mathcal{C}}}^{\phi(\Psi)} r^{\phi(\Psi)+\epsilon-\frac{2}{p}} \Psi(\mathcal{L}_O^{J-1}R)|_{p,S} \Big|_{C_0 \cup \underline{\mathcal{C}}_0} \leq \hat{C}_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)}}{\rho_{0,1}^J} \quad (4.17)$$

and finally

$$||\lambda|_{\underline{\mathcal{C}}}^{\phi(\Psi)} r^{\phi(\Psi)+\epsilon-\frac{2}{p}} \Psi(\hat{\mathcal{L}}_O^{J-1}R)|_{p,S} \Big|_{C_0 \cup \underline{\mathcal{C}}_0} \leq \hat{C}_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)(\nu)}}{\rho_{0,1}^J}, \quad (4.18)$$

Step 2: Once we have the estimates for $\Psi(\hat{\mathcal{L}}_O^{J-1}R)$ on $C_0 \cup \underline{C}_0$ we derive easily an estimate for $\mathcal{Q}_{(0)}^{(J-2)}(\lambda_0, \nu)$ and for $\underline{\mathcal{Q}}_{(0)}^{(J-2)}(\lambda, \nu_0)$ namely,

$$\mathcal{Q}_{(0)}^{(J-2)}(\lambda_0, \nu) + \underline{\mathcal{Q}}_{(0)}^{(J-2)}(\lambda, \nu_0) \leq \left(\hat{C}_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_{0,1}} e^{(J-2)\Gamma(\lambda)}}{\rho_{0,1}^J} \right)^2, \quad (4.19)$$

with $\delta_{0,1} > \delta_0$. The proof of estimate 4.19 is in Lemma 10.1.

Step 3: Once we control of the $\mathcal{Q}_{(0)}^{(J-2)}(\lambda_0, \nu)$ and the $\underline{\mathcal{Q}}_{(0)}^{(J-2)}(\lambda, \nu_0)$ norms, assuming the initial data sufficiently small we prove as discussed in the previous subsection and in Appendix 15 that, in the region where the a priori estimates hold for $J \leq 7$, we have for all J ,

$$\mathcal{Q}^{(J-2)}(C(\lambda; [\nu_0, \nu])) + \underline{\mathcal{Q}}^{(J-2)}(\underline{C}(\nu; [\lambda_0, \lambda])) \leq C^{(1)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta + \Gamma(\lambda))}}{\rho_{0,1}^J} \right) \quad (4.20)$$

where

$$C^{(1)} = c \left[\hat{C}_{0,0} + \left(\varepsilon + C C^{(1)} \frac{e^{-\delta}}{\rho_{0,1}} \right) \right] < \hat{C}_{0,0}.$$

This is the content of Theorem 10.1 proved in section 10. **Remark:**

The estimates of the $\mathcal{Q}^{(J-2)}$ norms is a substantially a generalization of chapter 6 of [Kl-Ni2] to the \mathcal{L}_O^{J-2} derivatives of the \mathcal{Q} norms, the proof is very involved, it requires to carefully estimate a great number of terms, It is here that the term $\Gamma(\lambda)$ plays its role to assure the boundedness of some important terms

Step 4: Proceeding again as in [Kl-Ni2], Chapter 5, with some more work, we control, using 4.20, the $\Psi(\hat{\mathcal{L}}_O^{J-1}R)$ norms obtaining

$$\|u|_{\underline{C}}^{\phi(\Psi)} r^{\phi(\Psi) - \frac{2}{p}} \Psi(\hat{\mathcal{L}}_O^{J-1}R)\|_{p,S}(\lambda, \nu) \leq C^{*(1)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)}}{\rho_{0,1}^J} \right). \quad (4.21)$$

The proof follows from Lemma 10.2 of the Arxiv version.

Step 5: First we prove that under the inductive assumptions for the connection coefficients norms up to order $N-1$ and the previous norm estimates for $\Psi(\hat{\mathcal{L}}_O^{J-1}R)$ with $J \leq N$ that the inequality hold, see section 15.5, lemmas 10.3 and 15.1:

$$\begin{aligned} \|u|_{\underline{C}}^{\phi(\Psi)} r^{\phi(\Psi) - \frac{2}{p}} \mathcal{L}_O^{J-1} \Psi\|_{p,S}(\lambda, \nu) &\leq C^{(1)} e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J}, \\ \|u|_{\underline{C}}^{\phi(\Psi)} r^{\phi(\Psi) - \frac{2}{p}} \nabla_O^{J-1} \Psi\|_{p,S}(\lambda, \nu) &\leq C^{(1)} e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J}, \end{aligned} \quad (4.22)$$

with $C^{*(1)} < C^{(1)}$. This is obtained substantially inverting lemma 4.2.

Remark:

Notice that in order to apply these estimates to the estimates for the connection coefficients the constant \tilde{C}_4 has to be smaller of the constants of C_i and \underline{C}_i of inequalities 3.27

Then we prove following estimates hold, for $\nabla^{J-1}\Psi(R)$ with $J \leq N$,

$$||u|^{\frac{\phi(\Psi)}{p}} r^{\phi(\Psi) - \frac{2}{p}} \nabla^{J-1}\Psi|_{p,S}(\lambda, \nu) \leq C^{(1)} e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho^J}, \quad (4.23)$$

with suitable $\rho < \rho_{0,1}$ and with $C^{(1)}$ of order $O(\varepsilon)$ for $J \leq 7$, of order $O(1)$ otherwise, smaller than the constants related to the estimates of the connection coefficients. This is the content of Lemma 10.4 of section 10, which completes the proof of Theorem 4.1 and whose proof is in the Appendix 15.

4.3 The \mathcal{D}_ν and \mathcal{D}_λ derivatives and the mixed ones for the null Riemann components

Also the mixed derivatives for the Riemann components have to be estimated through the \mathcal{Q} norms.

These estimates for the connection coefficients are needed as we remarked, see (I) for a more detailed discussion, that the recursive estimates for the angular derivatives of the connection coefficients are not sufficient to prove the existence of an extended region for the analytic solutions.⁶⁴ In fact on $\underline{C}(\nu)$ we have to control all the angular ∇ derivatives and all the \mathcal{D}_3 derivatives mixed together in all the possible ways. Analogously on the generic outgoing cone $C(\lambda)$ we have to control all the angular ∇ derivatives and all the \mathcal{D}_4 derivatives mixed together in all the possible ways.

In section 3.6, when we discussed how to control all the “tangential” derivatives of the connection coefficients, it follows that we have to control all the tangential derivatives of the Riemann components which require to prove that a larger set of energy type \mathcal{Q} norms, always made in terms of the Bell-Robinson tensor, is bounded; this is obtained proceeding in the same way as done for the angular derivatives in Section 4. The remaining \mathcal{Q} norms we have to introduce have, in fact, the same general structure. The basic difference is that the $\hat{\mathcal{L}}_O^{J-2}W$ present there has to be substituted by a sum of analogous terms with $\hat{\mathcal{L}}_S^p \hat{\mathcal{L}}_O^k W$ with $p + k \leq N - 2$, $p > 0$. Observe, as discussed later on, that although we have to control the general mixed derivatives, with arbitrary distribution of the various derivatives for the Riemann components, nevertheless it is enough to introduce the \mathcal{Q} norms relative to the Weyl tensors $\tilde{W}_{k,p} = \hat{\mathcal{L}}_S^k \hat{\mathcal{L}}_O^p W$. The boundedness of these \mathcal{Q} norms have to be proved exactly in the same way obtaining the exact analogous of the estimate 10.8, with $p + k = H$,

$$\int_{C(\lambda_0; [\nu_0, \nu])} Q(\hat{\mathcal{L}}_S^k \hat{\mathcal{L}}_O^p W)(\bar{K}, \bar{K}, T, e_4) \leq \left(\tilde{C}^{(1)} \frac{H!}{H^\alpha} \frac{e^{(H-2)\delta_0} e^{(H-2)\Gamma(\lambda)}}{\rho_{0,1}^H} \right)^2. \quad (4.24)$$

⁶⁴To obtain a Cauchy-Kowalevski solution starting with data on an inner outgoing or incoming cone we need to prove that they are analytic on them.

Once we control, via the “extended \mathcal{Q} norms” all the derivatives up to $N - 1$ of the null Riemann components we are able to control the $\mathcal{P}_{3,4}$ derivatives and the mixed ones for the connection coefficients up to order N .

Repeating the steps of the section 4, discussed in details in section 10 and appendix 15, but now for the $D_S \nabla_O$ derivatives and exploiting the smallness of mixed derivatives of the initial data, see Theorem 3.4, we can state the following theorem analogous to theorem 3.6,

Theorem 4.2 *Under the assumptions 3.22 and 3.23 on the mixed derivatives of the initial data connection coefficients on $C_0 \cup \underline{C}_0$, the following estimates hold on \mathcal{K} , with $\tilde{C}^{(1)}$ of order $O(\varepsilon)$ for any $J \leq 7$ and order $O(1)$ otherwise.*

$$||\lambda|^{\phi(\Psi)} r^{\phi(\Psi) - \frac{2}{p}} \Psi(\hat{\mathcal{L}}_S^p \hat{\mathcal{L}}_O^k W)|_{p,S}(\lambda, \nu) \leq \tilde{C}^{(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta + \Gamma(\lambda))}}{\rho_{0,1}^J} . \quad (4.25)$$

$$||\lambda|^{\phi(\Psi)} r^{\phi(\Psi) - \frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^k \Psi|_{p,S}(\lambda, \nu) \leq \tilde{C}^{(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta + \Gamma(\lambda))}}{\rho_{0,1}^J} . \quad (4.26)$$

The estimates of the $\mathcal{P}_\nu^J \nabla^P$ mixed derivatives of the Riemann components proceed in the same way of Lemma 3.1

Lemma 4.3 *Let us assume the estimates of theorem 4.2, then the null Riemann components Ψ satisfy the following estimates for $p + q = J$ on \mathcal{K} , with $\rho_1 < \rho < \rho_{0,1}$ depending only the first s derivatives of the initial data, with $s \leq 7$ and with a suitable choice of the $\tilde{C}^{(1)}$ constant of order $O(\varepsilon)$ for $J \leq 7^{65}$ and $O(1)$ otherwise.*

$$||\lambda|^{\phi(\Psi)} r^{\phi(\Psi) + k - \frac{2}{p}} D_S^p \nabla^k \Psi|_{p,S}(\lambda, \nu) \leq \tilde{C}^{(1)} e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_1^J} , \quad (4.27)$$

$$\begin{aligned} \phi(\alpha) = \phi(\beta) = \frac{7}{2} , \quad \phi(\rho) = \phi(\sigma) = 3 , \quad \phi(\underline{\beta}) = 2 , \quad \phi(\underline{\alpha}) = 1 \\ \underline{\phi}(\alpha) = \underline{\phi}(\beta) = 0 , \quad \underline{\phi}(\rho) = \underline{\phi}(\sigma) = \frac{1}{2} , \quad \underline{\phi}(\underline{\beta}) = \frac{3}{2} , \quad \underline{\phi}(\underline{\alpha}) = \frac{5}{2} . \end{aligned} \quad (4.28)$$

Proof: As in lemma 3.1, also in this case the proof is an adapted version of lemma 4.1 .

From these lemmas the following lemma holds,

⁶⁵If $J \geq 1$, for $J = 0$ ρ becomes $\rho - \underline{\rho}$, see [Kl-Ni2].

Lemma 4.4 *Under the assumptions of Lemma 4.3 the following estimates hold on \mathcal{K} , with $p + k = J$:*

$$\begin{aligned} ||\lambda|^{\phi(\Psi)} r^{\phi(\Psi)+(J-1)-\frac{2}{p}} D_\nu^p \nabla^k \Psi|_{p,S}(\lambda, \nu) &\leq \tilde{C}^{(1)} e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_2^p \rho_1^k} \\ ||\lambda|^{\phi(\Psi)+p} r^{\phi(\Psi)+k-\frac{2}{p}} D_\lambda^p \nabla^k \Psi|_{p,S}(\lambda, \nu) &\leq \tilde{C}^{(1)} e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_2^p \rho_1^k}. \end{aligned} \quad (4.29)$$

Proof: See subsection 15.7.

Remark:

Also in this case we can choose $\tilde{C}^{(1)}$ smaller than the constants related to the estimates of the mixed derivatives of the connection coefficients.

5 The estimates for the derivatives of the metric components

To complete the existence proof of the global analytic solution of the characteristic problems we are considering, we need to control also all the derivatives of the various metric components. In fact the system of first order equations we have to solve is,⁶⁶ omitting the indices to simplify the notations,

$$\begin{aligned} \frac{\partial \gamma}{\partial \omega} - v = 0 \quad , \quad \frac{\partial \log \Omega}{\partial \omega} - \psi = 0 \quad , \quad \frac{\partial \hat{X}}{\partial \omega} - w = 0 \\ \frac{\partial \gamma}{\partial \lambda} - 2\Omega \underline{\chi} + \mathcal{L}_X \gamma = 0 \\ \frac{\partial \log \Omega}{\partial \lambda} + \psi(X) + 2\Omega \underline{\omega} = 0 \\ \frac{\partial v}{\partial \lambda} + \nabla_X v + (\not\partial X) \cdot v - S(\not\partial \otimes w) - 2\Omega \not\partial \otimes \underline{\chi} - 2\Omega \psi \otimes \underline{\chi} = 0 \\ \frac{\partial \psi}{\partial \lambda} + \nabla_X \psi + 2\Omega \underline{\omega} \psi + \psi(\nabla X) + 2\Omega \not\partial \underline{\omega} = 0 \\ \frac{\partial \text{tr} \chi}{\partial \lambda} + \Omega \text{tr} \chi \text{tr} \chi - 2\Omega \underline{\omega} \text{tr} \chi + \nabla_X \text{tr} \chi - 2\Omega \text{div}(\zeta + \psi) - 2\Omega |\zeta + \psi|^2 + 2\Omega \mathbf{K} = 0 \\ \frac{\partial \hat{\chi}}{\partial \lambda} + \mathcal{L}_X \hat{\chi} - \frac{\Omega \text{tr} \chi}{2} \hat{\chi} + \frac{\Omega \text{tr} \chi}{2} \hat{\chi} - 2\Omega \underline{\omega} \hat{\chi} - \Omega(\hat{\chi} \cdot \hat{\chi}) \gamma - \Omega \nabla \hat{\otimes}(\zeta + \psi) - \Omega(\zeta + \psi) \hat{\otimes}(\zeta + \psi) = 0 \\ \frac{\partial \zeta}{\partial \lambda} + \mathcal{L}_X \zeta + \Omega \text{tr} \chi \zeta + \Omega \text{div} \hat{\chi} - \frac{1}{2} \Omega \not\partial \text{tr} \chi + 2\Omega \underline{\omega} \psi + 2\Omega \not\partial \underline{\omega} + \Omega \psi \cdot \underline{\chi} = 0 \\ \frac{\partial \omega}{\partial \lambda} + \not\partial_X \omega - 2\Omega \underline{\omega} \omega - \frac{3}{2} \Omega |\zeta|^2 + \frac{1}{4} \Omega \zeta \cdot \psi + \frac{1}{2} \Omega |\psi|^2 + \frac{1}{2} \Omega \left(\mathbf{K} + \frac{1}{4} \text{tr} \chi \text{tr} \chi - \frac{1}{2} \hat{\chi} \cdot \hat{\chi} \right) = 0 \end{aligned} \quad (5.1)$$

⁶⁶Remember that eqs. 5.1, 5.2 are first used to prove the existence of an analytic solution in a, possibly small, region, via Cauchy-Kovalevski as discussed in (I) and subsequently used again to extend the assumed maximal analyticity region once we have all the required norm estimates.

$$\begin{aligned}
& \frac{\partial \hat{X}}{\partial \nu} + 4\Omega^2 \zeta = 0 \\
& \frac{\partial w}{\partial \nu} + 8\Omega^2 \psi \otimes \zeta + 4\Omega^2 \not\partial \otimes \zeta - 2\Omega \psi \otimes (\chi \cdot X) - 2\Omega (\not\partial \otimes \chi) \cdot X = 0 \\
& \frac{\partial \text{tr} \underline{\chi}}{\partial \nu} + \Omega \text{tr} \chi \text{tr} \underline{\chi} - 2\Omega \omega \text{tr} \underline{\chi} + 2\Omega \text{div} \zeta - 2\Omega \text{div} \psi - 2\Omega |\zeta - \psi|^2 + 2\Omega \mathbf{K} = 0 \\
& \frac{\partial \hat{\chi}}{\partial \nu} - \frac{\Omega \text{tr} \chi}{2} \hat{\chi} + \frac{\Omega \text{tr} \underline{\chi}}{2} \hat{\chi} - 2\Omega \omega \hat{\chi} - \Omega (\hat{\chi} \cdot \hat{\chi}) \gamma + \Omega \nabla \widehat{\otimes} (\zeta - \psi) - \Omega (\zeta - \psi) \widehat{\otimes} (\zeta - \psi) = 0 \\
& \frac{\partial \underline{\omega}}{\partial \nu} - 2\Omega \omega \underline{\omega} - \frac{3}{2} \Omega |\zeta|^2 - \Omega \zeta \cdot \psi + \frac{1}{2} \Omega |\psi|^2 + \frac{1}{2} \Omega \left(\mathbf{K} + \frac{1}{4} \text{tr} \underline{\chi} \text{tr} \chi - \frac{1}{2} \hat{\chi} \cdot \hat{\chi} \right) = 0 .
\end{aligned} \tag{5.2}$$

Therefore, to complete our proof we need to control all the derivatives of γ_{ab}, Ω and X_a recalling that we are considering the “coordinate” components of these quantities. More precisely we need all the angular, $\not\partial$, and the ∂_ν derivatives for γ and Ω and the angular and ∂_λ derivatives for X . The proof of these estimates is in Section 11 .

5.1 The Ω component

The angular derivatives of this component are easy to control as

$$\not\partial \log \Omega = \nabla \log \Omega = 2^{-1}(\eta + \underline{\eta})$$

and η and $\underline{\eta}$ are already under control. The final result is,

$$|\mathcal{L}_O^{J-1} \partial \Omega|_{p,S} \leq c \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho^J} \right) . \tag{5.3}$$

5.2 The γ components

We recall that γ_{ab} satisfies the following equation

$$\frac{\partial}{\partial \nu} \gamma_{ab} - \Omega \text{tr} \chi \gamma_{ab} = 2\Omega \hat{\chi}_{ab} , \tag{5.4}$$

which we rewrite as

$$\frac{\partial}{\partial \nu} \gamma_{ab} - \overline{\Omega \text{tr} \chi} \gamma_{ab} = (\Omega \text{tr} \chi - \overline{\Omega \text{tr} \chi}) \gamma_{ab} + 2\Omega \hat{\chi}_{ab} \tag{5.5}$$

and

$$\frac{\partial}{\partial \nu} (r^{-2} \gamma_{ab}) = (\Omega \text{tr} \chi - \overline{\Omega \text{tr} \chi}) (r^{-2} \gamma_{ab}) + 2\Omega r^{-2} \hat{\chi}_{ab} .$$

The control of the angular non covariant derivatives can be easily obtained applying $\not\partial^J$ to both sides. The following estimates hold

$$|r^{J-2-\frac{2}{p}} \mathcal{L}_O^{J-1} \partial \gamma_{ab}|_{p,S}(\lambda, \nu) \leq c \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho^J} \right) . \tag{5.6}$$

Remark:

The estimates for $\partial^J \gamma_{ab}$ follow with $J!$ instead of $(J-1)!$ due to the loss of derivatives present in the transport equation 5.4. This loss is not present considering the mixed derivatives.

5.3 The X^a components

We study the equivalent $X_a = \gamma_{ac} X^c$ components; we have the transport equation

$$\frac{\partial X_a}{\partial \nu} = -4\Omega^2 \zeta_a . \quad (5.7)$$

Considering the norm $|X| = \sqrt{\gamma_{ab} X^a X^b}$ after a somewhat cumbersome calculation we obtain the final estimate for all J ,⁶⁷

$$\left| \frac{r^{(J+1-\frac{2}{p})}}{\log r} \mathcal{L}_O^{J-1} \partial |X| \right|_{p,S}(\lambda, \nu) \leq c \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho^J} ,$$

6 The control of the non covariant partial derivatives of the connection coefficients

Assume for simplicity \mathcal{O} denotes a S -tangent vector, as for instance the connection coefficient ζ , we are interested to the analyticity of the tensor field $\zeta = \zeta(e_C) \theta^C(\cdot)$ which means that we have to prove that the various components $\zeta(e_C)(\lambda, \nu, \omega^a)$ are analytic functions in the λ, ν, ω^a variables. To prove it we have to control the (norms of the) mixed derivatives in ν, ω^a for the quantities defined on an outgoing cone and the mixed derivatives in λ, ω^a for the quantities defined on an incoming cone. The proof of this result is in Section 12 .

7 The initial data for the global extension

In the inductive proofs of the previous sections we implicitly assumed that we can assign analytic initial data on the whole $C_0 \cup \underline{C}_0$ hypersurface satisfying appropriate smallness conditions. This requires a careful discussion and the goal of the present section is to show how this has to be done.

The problem we have to confront with is the following one: to prove Theorem 3.1 we use for the non underlined connection coefficients, \mathcal{O} ,⁶⁸ the transport

⁶⁷With some extra work it is possible to prove that,

$$\left| \frac{r^{J-\frac{2}{p}}}{\log r} \mathcal{L}_O^{J-1} \partial X_a \right|_{p,S} \leq c \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho^J} \right) .$$

⁶⁸With the exception of ω and $\underline{\omega}$ whose roles are inverted.

equations of these quantities along the outgoing cones starting from their values on \underline{C}_0 , and the opposite for the underlined quantities \underline{Q} .⁶⁹ This requires that we have analytic initial data for \mathcal{O} on \underline{C}_0 and analytic initial data for \underline{Q} on C_0 with their $L_{p,S}$ norms satisfying the estimates 3.31, 3.32. As, on the other side, the Cauchy Kowalevski theorem we use for the local solution and in the extension proof, see (I), requires also on C_0 analytic initial data for \mathcal{O} and the same for \underline{Q} on \underline{C}_0 , the conclusion is that all the connection coefficients have to be analytically assigned on C_0 and on \underline{C}_0 with their tangent derivatives satisfying the norm bounds 3.31, 3.32.

Moreover, as previously remarked, the initial conditions on $C_0 \cup \underline{C}_0$ for the connection coefficients have to be such that on these initial cones the Q norms are bounded. This requires that the null Riemann components, which are expressed in terms of the connection coefficients, have the appropriate decay and this, at its turn, requires for some of the connection coefficients a decay, on $C_0 \cup \underline{C}_0$, stronger than the one which will be proved in the internal region. To fulfill it we recall, first of all, the decay conditions required to the null Riemann components to have the $\mathcal{Q}_0^{(J-2)}$ norms finite and $O(\varepsilon)$ for $J = 2$; denoting with ∇ both tangential and normal derivatives,⁷⁰ we require, for $J < 3$,

$$\begin{aligned} |r^{\frac{7}{2}+J+\epsilon-\frac{2}{p}} \nabla^J \alpha|_{p=2,S} &\leq \varepsilon \quad , \quad |r^{\frac{7}{2}+J+\epsilon-\frac{2}{p}} \nabla^J \beta|_{p=2,S} \leq \varepsilon \quad , \\ |\lambda_0|^{\frac{1}{2}} r^{3+J-\frac{2}{p}} \nabla^J (\rho - \bar{\rho}, \sigma)|_{p=2,S} &\leq \varepsilon \quad , \quad |\lambda_0|^{\frac{3}{2}} r^{2+J+\epsilon-\frac{2}{p}} \nabla^J \underline{\beta}|_{p=2,S} \leq \varepsilon \end{aligned} \quad (7.1)$$

and same bounds for $J \geq 3$, with ε substituted by $C^{(0)} = O(1)$. The analogous asymptotic conditions required along \underline{C}_0 , to guarantee that also the \underline{Q}_0 norms are bounded are, again for $J < 3$,

$$\begin{aligned} |r^{\frac{7}{2}+J+\epsilon-\frac{2}{p}} \nabla^J \beta|_{p=2,S} &\leq \varepsilon \quad , \quad ||\lambda|^{\frac{1}{2}} r^{3+J-\frac{2}{p}} \nabla^J (\rho - \bar{\rho}, \sigma)|_{p=2,S} \leq \varepsilon \quad , \\ ||\lambda|^{\frac{3}{2}+\epsilon} r^{2+J-\frac{2}{p}} \nabla^J \underline{\beta}|_{p=2,S} &\leq \varepsilon \quad , \quad ||\lambda|^{\frac{5}{2}+\epsilon} r^{1+J-\frac{2}{p}} \nabla^J \underline{\alpha}|_{p=2,S} \leq \varepsilon \end{aligned} \quad (7.2)$$

and same bounds for $J \geq 3$, with ε substituted by $C^{(0)} = O(1)$.

Differently from what we do in the internal region, on $C_0 \cup \underline{C}_0$ the null Riemann components are directly estimated in terms of the connection coefficients and their first derivatives, this implies that the initial data connection coefficients have to be such that conditions 7.1 and 7.2 are satisfied.

Due to the constraints for the initial data one has to prove that these initial data can be consistently defined on the whole $C_0 \cup \underline{C}_0$. This could be a difficulty looking at their transport equations on C_0 (the same argument holds on \underline{C}_0 and we do not repeat it here), as, while in the transport equations for the not underlined quantities there is not any “loss of derivatives”, the opposite happens for the underlined ones. The problem is solved observing that the construction

⁶⁹The problem is even more delicate due to the fact that we have, for the non underlined quantities, to define “initial data” on the last slice, see the discussion in subsection 9.5.

⁷⁰The factor $|\lambda_0|$ constant on C_0 is, of course, irrelevant here and left just to remind the expected behavior in the interior.

of these initial data is done in a well defined order; in other words there is a natural order in the use of the different structure equations which allows to construct the initial data without never facing with the “loss of derivatives” problem. This is the content of the argument we discuss in detail in Section 13, already exploited in [Ca-Nil] referring only to the first few derivatives, implying that we have to choose $J_0 = 7$. Here we state our final result concerning the initial data,

Theorem 3.2 Assuming $\hat{\chi}$ on C_0 , $\underline{\hat{\chi}}$ and X on \underline{C}_0 and Ω on $C_0 \cup \underline{C}_0$ satisfying the following estimates,

$$\begin{aligned} |r^{J+2+\epsilon-\frac{2}{p}} \nabla^J \log \Omega|_{p,S}(\lambda_0, \nu) &\leq C_3^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,0,1}^J} \\ |r^{J+\frac{5}{2}+\epsilon-\frac{2}{p}} \nabla^J \hat{\chi}|_{p,S}(\lambda_0, \nu) &\leq C_1^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,0,1}^J}, \quad (7.3) \\ ||\lambda|^{1+\epsilon} r^{1+J-\frac{2}{p}} \nabla^J \log \Omega|_{p,S}(\lambda, \nu_0) &\leq \underline{C}_3^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,0,1}^J} \\ ||\lambda|^{\frac{3}{2}+\epsilon} r^{1+J-\frac{2}{p}} \nabla^J \underline{\hat{\chi}}|_{p,S}(\lambda, \nu_0) &\leq \underline{C}_1^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,0,1}^J}, \end{aligned}$$

with $C_{1,3}^{(0)}$ and $\underline{C}_{1,3}^{(0)}$ of order $O(\varepsilon)$ for $J \leq 7$ and $O(1)$ otherwise, assuming $\underline{\omega}$ have some definite expressions on S_0 depending on the remaining connection coefficients on S_0 , see [Ca-Nil], assuming, finally, that on S_0 the following conditions are satisfied^{71,72},

$$\begin{aligned} \nabla \text{tr} \chi - \zeta \text{tr} \chi &= O(\varepsilon r_0^{-(\frac{7}{2}+\epsilon)}) \\ \nabla \text{tr} \underline{\chi} - \zeta \text{tr} \underline{\chi} &= O(\varepsilon r_0^{-(\frac{7}{2}+\epsilon)}) \\ \mathbf{K} - \overline{\mathbf{K}} + \frac{1}{4}(\text{tr} \chi \text{tr} \underline{\chi} - \overline{\text{tr} \chi \text{tr} \underline{\chi}}) &= O(\varepsilon r_0^{-\frac{7}{2}}) \\ \text{curl} \zeta &= O(\varepsilon r_0^{-\frac{7}{2}}). \end{aligned} \quad (7.4)$$

then it is possible to construct analytic initial data on $C_0 \cup \underline{C}_0$ such that the energy type \mathcal{Q}^0 norms on $C_0 \cup \underline{C}_0$ are finite and small and the connection coefficients norm satisfy the following estimates, with $\rho_0 < \rho_{0,0,1}$ and the constant $C_{(J)}$ of order $O(\varepsilon)$ for $J \leq 7$ and $O(1)$ otherwise,

On C_0 :

$$|r^{1+J+\sigma(J)-\frac{2}{p}} \nabla^J \text{tr} \chi|_{p,S}(\lambda_0, \nu_0) \leq C_0^{(0)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^J} \right)$$

⁷¹To completely define the ∇ derivative on S_0 we have also to assign γ on S_0 see section 13.1.1

⁷²Clearly, to obtain the smallness of the first $C_{(J)}$ $J \leq 7$, we have to assume also the angular derivatives ∇^s of the quantities assigned on S_0 of order ε .

$$\begin{aligned}
|r^{\frac{5}{2}+J+\epsilon-\frac{2}{p}} \nabla^J \hat{\chi}|_{p,S}(\lambda_0, \nu_0) &\leq C_1^{(0)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^J} \right) \\
|r^{2+J-\frac{2}{p}} \nabla^J \zeta|_{p,S}(\lambda_0, \nu_0) &\leq C_4^{(0)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^J} \right) \\
|r^{2+J+\epsilon-\frac{2}{p}} \nabla^J \omega|_{p,S}(\lambda_0, \nu_0) &\leq C_2^{(0)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^J} \right) \\
|r^{1+J+\sigma(J)-\frac{2}{p}} \nabla^J \text{tr} \underline{\chi}|_{p,S}(\lambda_0, \nu) &\leq C_5^{(0)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^J} \right) \\
|\lambda_0| r^{1+J-\frac{2}{p}} \nabla^J \hat{\underline{\chi}}|_{p,S}(\lambda_0, \nu) &\leq C_6^{(0)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^J} \right) \\
|\lambda_0| r^{1+J-\frac{2}{p}} \nabla^J \underline{\omega}|_{p,S}(\lambda_0, \nu) &\leq C_7^{(0)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^J} \right)
\end{aligned} \tag{7.5}$$

On \underline{C}_0 :

$$\begin{aligned}
|\lambda|^{|\sigma(J)|} r^{J+1-\frac{2}{p}} \nabla^J \text{tr} \underline{\chi}|_{p,S} &\leq \underline{C}_0^{(0)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^J} \right) \\
|\lambda|^{\frac{3}{2}+\epsilon} r^{J+1-\frac{2}{p}} \nabla^J \hat{\underline{\chi}}|_{p,S} &\leq \underline{C}_1^{(0)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^J} \right) \\
|\lambda|^{1+\epsilon} r^{J+1-\frac{2}{p}} \nabla^J \underline{\omega}|_{p,S} &\leq \underline{C}_2^{(0)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^J} \right) \\
|r^{1+J+\sigma(J)-\frac{2}{p}} \nabla^J \text{tr} \chi|_{p,S} &\leq \underline{C}_5^{(0)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^J} \right) \\
|r^{J+2-\frac{2}{p}} \nabla^J \hat{\chi}|_{p,S} &\leq \underline{C}_6^{(0)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^J} \right) \\
|r^{J+2-\frac{2}{p}} \nabla^J \omega|_{p,S} &\leq \underline{C}_7^{(0)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^J} \right) \\
|r^{J+2-\frac{2}{p}} \nabla^J \zeta|_{p,S} &\leq \underline{C}_4^{(0)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^J} \right) .
\end{aligned} \tag{7.6}$$

The condition for $\underline{\omega}$ on S_0 is

$$\underline{\omega}(\nu_0) = - \int_{\nu_0}^{\infty} d\nu' e^{\int_{\nu_0}^{\nu'} (-2\Omega\omega) d\nu''} \left[\zeta \cdot \nabla \log \Omega + \frac{3}{2} |\zeta|^2 - \frac{1}{2} |\nabla \log \Omega|^2 - \frac{1}{2} (\mathbf{K} + \frac{1}{4} \text{tr} \chi \text{tr} \underline{\chi} - \frac{1}{2} \hat{\chi} \cdot \hat{\underline{\chi}}) \right] (\nu') . \tag{7.7}$$

These are the estimates 3.15 and 3.16, in order to prove estimates 3.17, 3.18, 3.19 and 3.20 we only have to reproduce the proofs of lemmas 4.1 and 4.2, with the obvious modifications.

Now we have to prove the estimates for the mixed derivatives of the connection coefficients on the initial data.

Theorem 3.4: *Assuming the hypothesis of Theorem 3.2, the estimates 3.21 for the $\mathbf{D}_\nu^J \nabla^P$ derivatives, then the following estimates hold for any J and P on*

$C_0 \cup \underline{C}_0$ where the constants F_i and \underline{F}_i are of order $O(\varepsilon)$ for $J \leq 7$ and $O(1)$ otherwise and $\sigma(P) = 0$ for $P = 0$ and $\sigma(P) = 1$ for $P > 0$.

$$\begin{aligned}
|r^{1+J+P+\sigma(P)-\frac{2}{p}} \mathfrak{D}_\nu^J \nabla^P U|_{p,S} &\leq F_1 \left(\frac{(J+P)!}{(J+P)^\alpha} \frac{e^{(J+P-2)(\delta+\underline{\Gamma}_0(\lambda))}}{\rho_0^{J+P}} \right) \\
|r^{1+J+P+\sigma(P)-\frac{2}{p}} \mathfrak{D}_\nu^J \nabla^P \text{tr} \chi|_{p,S} &\leq F_2 \left(\frac{(J+P)!}{(J+P)^\alpha} \frac{e^{(J+P-2)(\delta+\underline{\Gamma}_0(\lambda))}}{\rho_0^{J+P}} \right) \\
|r^{2+J+P-\frac{2}{p}} \mathfrak{D}_\nu^J \nabla^P \hat{\chi}|_{p,S} &\leq F_3 \left(\frac{(J+P)!}{(J+P)^\alpha} \frac{e^{(J+P-2)(\delta+\underline{\Gamma}_0(\lambda))}}{\rho_0^{J+P}} \right) \\
|r^{2+J+P-\frac{2}{p}} \mathfrak{D}_\nu^J \nabla^P \eta|_{p,S} &\leq F_4 \left(\frac{(J+P)!}{(J+P)^\alpha} \frac{e^{((J+P)-2)(\delta+\underline{\Gamma}_0(\lambda))}}{\rho_0^{J+P}} \right) \\
|r^{2+J+P-\frac{2}{p}} \mathfrak{D}_\nu^J \nabla^P \zeta|_{p,S} &\leq F_5 \left(\frac{(J+P+1)!}{(J+P+1)^\alpha} \frac{e^{((J+P)-1)(\delta+\underline{\Gamma}_0(\lambda))}}{\rho_0^{J+P+1}} \right) \\
|r^{2+J+P-\frac{2}{p}} \mathfrak{D}_\nu^J \nabla^P \underline{U}|_{p,S} &\leq \underline{F}_1 \left(\frac{(J+P)!}{(J+P)^\alpha} \frac{e^{((J+P)-2)(\delta+\underline{\Gamma}_0(\lambda))}}{\rho_0^{J+P}} \right) \\
|r^{1+J+P+\sigma(P)-\frac{2}{p}} \mathfrak{D}_\nu^J \nabla^P \text{tr} \underline{\chi}|_{p,S} &\leq \underline{F}_2 \left(\frac{(J+P)!}{(J+P)^\alpha} \frac{e^{(J+P-2)(\delta+\underline{\Gamma}_0(\lambda))}}{\rho_0^{J+P}} \right) \\
||\lambda| r^{1+J+P-\frac{2}{p}} \mathfrak{D}_\nu^J \nabla^P \underline{\hat{\chi}}|_{p,S} &\leq \underline{F}_3 \left(\frac{(J+P)!}{(J+P)^\alpha} \frac{e^{((J+P)-2)(\delta+\underline{\Gamma}_0(\lambda))}}{\rho_0^{J+P}} \right) \\
|r^{2+J+P-\frac{2}{p}} \mathfrak{D}_\nu^J \nabla^P \omega|_{p,S} &\leq F_6 \left(\frac{(J+P+1)!}{(J+P+1)^\alpha} \frac{e^{((J+P)-1)(\delta+\underline{\Gamma}_0(\lambda))}}{\rho_0^{J+P+1}} \right) \\
|r^{2+J+P-\frac{2}{p}} \mathfrak{D}_\nu^J \nabla^P \underline{\omega}|_{p,S} &\leq \underline{F}_6 \left(\frac{(J+P)!}{(J+P)^\alpha} \frac{e^{((J+P)-2)(\delta+\underline{\Gamma}_0(\lambda))}}{\rho_0^{J+P}} \right).
\end{aligned} \tag{7.8}$$

Proof: First of all notice that the hypothesis of Theorem 3.2, already provide the right estimates for $\mathfrak{D}_\nu^J \nabla^P \chi$ on C_0 , $\mathfrak{D}_\nu^J \nabla^P \underline{\chi}$ and $\mathfrak{D}_\nu^J \nabla^P X$ on \underline{C}_0 , $\mathfrak{D}_\nu^J \nabla^P \Omega$ and hence for $\mathfrak{D}_\nu^J \nabla^P \omega$, $\mathfrak{D}_\nu^J \nabla^P \underline{\omega}$ on $C_0 \cup \underline{C}_0$, for any ν and P . For the other connection coefficients the proof is recursive, we assume these estimates hold for for (J, P) and we prove they hold also for for $(J+1, P-1)$, with $J+P = N$. We sketch the idea of the proof for $U = \Omega^{-1} \text{tr} \chi$ and leave the details in Appendix 13.

First of all we observe that the basis of the induction is exactly the result of Theorem 3.2. Hence, recalling the transport equation

$$\mathfrak{D}_\nu U + \frac{\Omega \text{tr} \chi}{2} U + |\hat{\chi}|^2 = 0, \tag{7.9}$$

we have the following expression

$$\mathfrak{D}_\nu^J \nabla^P U = \mathfrak{D}_\nu^{J-1} \nabla^P \mathfrak{D}_\nu U + \mathfrak{D}_\nu^{J-1} [\mathfrak{D}_\nu, \nabla^P] U = \mathfrak{D}_\nu^{J-1} \nabla^P \left(-\Omega \frac{\text{tr} \chi}{2} U - |\hat{\chi}|^2 \right) + \mathfrak{D}_\nu^{J-1} [\mathfrak{D}_\nu, \nabla^P] U$$

$$\begin{aligned}
&= -\frac{1}{2} \mathfrak{D}_\nu^{J-1} \nabla^P (\Omega \text{tr} \chi U) - \mathfrak{D}_\nu^{J-1} \nabla^P (|\hat{\chi}|^2) + \mathfrak{D}_\nu^{J-1} [\mathfrak{D}_\nu, \nabla^P] U \\
&= -\frac{1}{2} \sum_{q=0}^{J-1} \sum_{h=0}^P \binom{J-1}{q} \binom{P}{h} (\mathfrak{D}_\nu^q \nabla^h \Omega \text{tr} \chi) (\mathfrak{D}_\nu^{J-1-q} \nabla^{P-h} U) - \sum_{q=0}^{J-1} \sum_{h=0}^P \binom{J-1}{q} \binom{P}{h} (\mathfrak{D}_\nu^q \nabla^h \hat{\chi}) \cdot (\mathfrak{D}_\nu^{J-1-q} \nabla^{P-h} \hat{\chi}) \\
&\quad + \mathfrak{D}_\nu^{J-1} ([\mathfrak{D}_\nu, \nabla^P] U) .
\end{aligned} \tag{7.10}$$

As $[\nabla^P, \frac{\mathcal{D}}{\partial \nu}] f = \sum_{k=0}^{P-1} \nabla^k [\nabla, \frac{\mathcal{D}}{\partial \nu}] \nabla^{P-k-1} f$ and from equation 9.1,

$$[\nabla_\mu, \frac{\mathcal{D}}{\partial \nu}] U_{\nu_1 \dots \nu_k} = - \sum_{j=1}^k C_{\mu \nu_j}^{\sigma_j} U_{\nu_1 \dots \sigma_j \dots \nu_k} + \Omega \chi_\mu^\rho (\nabla_\rho U)_{\nu_1 \dots \nu_k} ,$$

it follows

$$\begin{aligned}
[\nabla^P, \frac{\mathcal{D}}{\partial \nu}] f &= \sum_{k=0}^{P-1} \nabla^k \left(-(P-k-1) C \nabla^{P-k-1} f + \Omega \chi \nabla^{P-k} f \right) \\
&= - \sum_{k=0}^{P-1} (P-k-1) \sum_{J=0}^k \binom{k}{J} (\nabla^J C) \nabla^{P-1-J} f + \sum_{k=0}^{P-1} \sum_{J=0}^k \binom{k}{J} (\nabla^J \Omega \chi) \nabla^{P-J} f .
\end{aligned} \tag{7.11}$$

Therefore we can write

$$\begin{aligned}
\mathfrak{D}_\nu^J \nabla^P U &= -\frac{1}{2} \sum_{q=0}^{J-1} \sum_{h=0}^P \binom{J-1}{q} \binom{P}{h} (\mathfrak{D}_\nu^q \nabla^h \Omega \text{tr} \chi) (\mathfrak{D}_\nu^{J-1-q} \nabla^{P-h} U) \\
&\quad - \sum_{q=0}^{J-1} \sum_{h=0}^P \binom{J-1}{q} \binom{P}{h} (\mathfrak{D}_\nu^q \nabla^h \hat{\chi}) \cdot (\mathfrak{D}_\nu^{J-1-q} \nabla^{P-h} \hat{\chi}) \\
&\quad + \sum_{q=0}^{J-1} \binom{J-1}{q} \sum_{k=0}^{P-1} (P-k-1) \sum_{j=0}^k \binom{k}{j} (\mathfrak{D}_\nu^q \nabla^j C) \mathfrak{D}_\nu^{j-1-q} \nabla^{P-1-j} U \\
&\quad - \sum_{q=0}^{J-1} \sum_{k=0}^{P-1} \sum_{j=0}^k \binom{J-1}{q} \binom{k}{j} (\mathfrak{D}_\nu^q \nabla^j \Omega \chi) \mathfrak{D}_\nu^{j-1-q} \nabla^{P-j} U
\end{aligned} \tag{7.12}$$

and using the inductive assumptions we can estimate the norms of the right hand side of 7.12 to prove the expected result.

8 Conclusions

We can therefore consider totally solved the global characteristic problem associated to two intersecting null cones; by this we mean that we can provide a global initial data set, specifying which quantities can be assigned freely, which ones have to be given on the intersection of the two cones and finally which constraints they have to satisfy on it. Moreover, we provide, the smallness conditions in the appropriate Sobolev norms they have to satisfy to obtain a weak global solution or, alternatively, given an analytic set of initial data, an analytic global solution. The future program will be to extend this result to the full null cone, extending to the tip the outgoing one and reducing consequently the ingoing cone to a point. This process presents some difficulties, the main ones we can foresee are:

- i) As the surface intersection of the two null hypersurfaces reduces to a point, how do the constrained quantities have to be assigned on it? How to reduce the constraint equations?
- ii) How the “lapse” function Ω has to be assigned in order to assure the orthogonal null geodesics starting from the $\nu = \text{constant}$ surfaces on the initial data outgoing cone intersect exactly in one point, so to form a null cone ?

We are confident that both questions, and other ones that can possibly emerge can find a “natural solution” in this formalism, allowing to extend the global characteristic problem to the full null cone.

9 The complete results of Section 3

9.1 The control of the angular derivatives of all the connection coefficients, the proof of Theorem 3.6

9.1.1 The control of $\mathcal{L}_O^{N-1}\psi$

First of all we state some commutation relations.

Lemma 9.1 *The following relations hold, denoting $\mathcal{D}_\nu := \Omega\mathcal{D}_4$,*

$$[\nabla_\mu, \frac{\mathcal{D}}{\partial\nu}]U_{\nu_1\dots\nu_k} = -\sum_{j=1}^k C_{\mu\nu_j}^{\sigma_j} U_{\nu_1\dots\sigma_j\dots\nu_k} + \Omega\chi_\mu^\rho(\nabla_\rho U)_{\nu_1\dots\nu_k} \quad (9.1)$$

$$C_{\mu\nu_j}^{\sigma_j} = \Omega\left[(\chi_{\mu\nu_j}\eta^{\sigma_j} - \chi_\mu^{\sigma_j}\eta_{\nu_j}) + \theta_\mu^C\theta_{\nu_j}^D R^{\sigma_j}(\cdot, e_C, e_4, e_D)\right] \quad (9.2)$$

in the case of U a 1-form:

$$\begin{aligned} [\nabla_\mu, \frac{\mathcal{D}}{\partial\nu}]U_\nu &= \\ &= \left[(\chi_{\mu\nu}\eta^\sigma - \chi_\mu^\sigma\eta_\nu) + \theta_\mu^C\theta_\nu^D R^\sigma(\cdot, e_C, e_4, e_D)\right]U_\sigma + \Omega\chi_\mu^\rho(\nabla_\rho U)_\nu \\ &= -C_{\mu\nu}^\sigma U_\sigma + \Omega\chi_\mu^\rho(\nabla_\rho U)_\nu \end{aligned} \quad (9.3)$$

We rewrite symbolically the last formula as:

$$[\nabla_\mu, \frac{\mathcal{D}}{\partial\nu}]U = -CU + D(\nabla U) \quad (9.4)$$

Where with C we mean a combination of null Riemann components and with D we mean a combination of connection coefficients and metric components.

Similarly we have:

$$\begin{aligned} [\nabla_\mu, \frac{\mathcal{D}}{\partial\lambda}]U_{\nu_1\dots\nu_k} &= \\ &= -\Omega\sum_{j=1}^k \left[(\chi_{\mu\nu_j}\eta^{\sigma_j} - \chi_\mu^{\sigma_j}\eta_{\nu_j}) + \theta_\mu^C\theta_{\nu_j}^D R^{\sigma_j}(\cdot, e_C, e_3, e_D)\right]U_{\nu_1\dots\sigma_j\dots\nu_k} + \Omega\chi_\mu^\rho(\nabla_\rho U)_{\nu_1\dots\nu_k} \\ &= -\sum_{j=1}^k C_{\mu\nu_j}^{\sigma_j} U_{\nu_1\dots\sigma_j\dots\nu_k} + \Omega\chi_\mu^\rho(\nabla_\rho U)_{\nu_1\dots\nu_k} \end{aligned} \quad (9.5)$$

which in the case of U 1-form becomes:

$$\begin{aligned} [\nabla_\mu, \frac{\mathcal{D}}{\partial\lambda}]U_\nu &= \\ &= \left[(\chi_{\mu\nu}\eta^\sigma - \chi_\mu^\sigma\eta_\nu) + \theta_\mu^C\theta_\nu^D R^\sigma(\cdot, e_C, e_3, e_D)\right]U_\sigma + \Omega\chi_\mu^\rho(\nabla_\rho U)_\nu \\ &= -\underline{C}_{\mu\nu}^\sigma U_\sigma + \Omega\chi_\mu^\rho(\nabla_\rho U)_\nu \end{aligned} \quad (9.6)$$

Also in this case we rewrite the last formula symbolically as

$$[\nabla_\mu, \frac{\mathcal{D}}{\partial\lambda}]U = -\underline{C}U + \underline{D}(\nabla U) \quad (9.7)$$

We have for the commutator of the tangential derivatives:

$$\begin{aligned} & [\nabla_\alpha, \nabla_\mu]U_{\nu_1\nu_2\dots\nu_q} = \\ & = \left[(\theta_\mu^3 \underline{\chi}_\alpha^\beta + \theta_\mu^4 \chi_\alpha^\beta) - (\theta_\alpha^3 \underline{\chi}_\mu^\beta + \theta_\alpha^4 \chi_\mu^\beta) \right] \nabla_\beta U_{\nu_1\nu_2\dots\nu_q} - \theta_\alpha^C \theta_\mu^D \sum_{i=1}^q R^{\tilde{\nu}_i}(\cdot, e_C, \cdot, e_D)_{\nu_i} U_{\nu_1\nu_2\dots\tilde{\nu}_i\dots\nu_q} \end{aligned} \quad (9.8)$$

In this case we apply this commutator to a 2-form obtaining:

$$[\nabla_\alpha, \nabla_\mu]U_{\nu_1\nu_2} = \quad (9.9)$$

$$\begin{aligned} & = \left[(\theta_\mu^3 \underline{\chi}_\alpha^\beta + \theta_\mu^4 \chi_\alpha^\beta) - (\theta_\alpha^3 \underline{\chi}_\mu^\beta + \theta_\alpha^4 \chi_\mu^\beta) \right] \nabla_\beta U_{\nu_1\nu_2} - \theta_\alpha^C \theta_\mu^D R^{\tilde{\nu}}(\cdot, e_C, \cdot, e_D)_{\nu_1} U_{\tilde{\nu}\nu_2} \\ & \quad - \theta_\alpha^C \theta_\mu^D R^{\tilde{\nu}}(\cdot, e_C, \cdot, e_D)_{\nu_2} U_{\nu_1\tilde{\nu}} \end{aligned} \quad (9.10)$$

Which we rewrite symbolically as

$$[\nabla_\mu, \frac{\mathcal{D}}{\partial\nu}]U = -\hat{C}U + \hat{D}(\nabla U) \quad (9.11)$$

At last we apply the $[\nabla_\alpha, \mathcal{D}iv]$ to a 2-form:

$$\begin{aligned} & [\nabla_\alpha, \mathcal{D}iv]U_{\nu_1\nu_2} = [\nabla_\alpha, \nabla_\mu]U_\nu^\mu = \\ & = \left[(\theta_\mu^3 \underline{\chi}_\alpha^\beta + \theta_\mu^4 \chi_\alpha^\beta) - (\theta_\alpha^3 \underline{\chi}_\mu^\beta + \theta_\alpha^4 \chi_\mu^\beta) \right] \nabla_\beta U_\nu^\mu - 2\theta_\alpha^C \theta_\mu^D R^{\tilde{\nu}}(\cdot, e_C, \cdot, e_D)_\nu U_\nu^\mu . \end{aligned} \quad (9.12)$$

Which we rewrite symbolically as

$$[\nabla_\alpha, \mathcal{D}iv]U = -\overline{C}U + \overline{D}(\nabla U) . \quad (9.13)$$

Notice that all the commutators have the same structure, namely a combination of products of connection coefficients and metric components times U and a combination of null Riemann components times ∇U .

Moreover the following relations hold for the commutator of the \mathcal{L}_O derivatives: For U 1-form

$$[\frac{\mathcal{D}}{\partial\nu}, \mathcal{L}_O]U = \left((O^\sigma C_{\sigma\mu}^\tau) + \frac{\mathcal{D}}{\partial\nu} \mathcal{H}_\mu^\tau \right) U_\tau$$

which we write in a compact way as

$$[\frac{\mathcal{D}}{\partial\nu}, \mathcal{L}_O]U = E_\mu^\tau U_\tau, \quad (9.14)$$

where

$$E_\mu^\tau \equiv \left((O^\sigma C_{\sigma\mu}^\tau) + \frac{\mathcal{D}}{\partial\nu} \mathcal{H}_\mu^\tau \right).$$

Where \mathcal{H} has the following expression

$$\mathcal{H}_{ab} = \mathbf{g}(\nabla_a O, \frac{\partial}{\partial x^b}) \quad (9.15)$$

With similar calculations we obtain

$$[\frac{\mathcal{D}}{\partial\lambda}, \mathcal{L}_O]U = \underline{E}_\mu^\tau U_\tau, \quad (9.16)$$

where

$$\underline{E}_\mu^\tau \equiv \left((O^\sigma \underline{C}_{\sigma\mu}^\tau) + \frac{\mathcal{D}}{\partial\lambda} \mathcal{H}_\mu^\tau \right).$$

Finally we consider the commutators $[\nabla, \mathcal{L}_O]U_{\alpha\beta}$ and $[\mathcal{D}iv, \mathcal{L}_O]U_{\alpha\beta}$ with $U_{\alpha\beta}$ a symmetric 2-form.

$$\begin{aligned} & ([\nabla_\nu, \mathcal{L}_O]U)_{ab} \\ &= O^\rho \hat{C}_{\rho\nu a}^\sigma U_{\sigma b} + 2U_{a\rho} (\nabla_\nu \mathcal{H}_b^\rho) + \sum_{A=1,2} (O^\rho \mathcal{H}_\rho^\beta \theta_\nu^A \nabla_\beta U_{ab}) \end{aligned} \quad (9.17)$$

which we rewrite symbolically as:

$$([\nabla_\nu, \mathcal{L}_O]U)_{ab} = \hat{E}_{\nu a}^\sigma U_{\sigma b} + \sum_{A=1,2} \hat{\mathcal{H}}_\nu^{A\beta} \nabla_\beta U_{ab} \quad (9.18)$$

and consequently

$$\begin{aligned} & ([\mathcal{D}iv, \mathcal{L}_O]U)_c = [\nabla_\nu, \mathcal{L}_O]U_c^\nu \\ &= O^\rho \hat{C}_{\rho\nu c}^\sigma U_\sigma^\nu + 2U_\rho^\nu (\nabla_\nu \mathcal{H}_c^\rho) + \sum_{A=1,2} (O^\rho \mathcal{H}_\rho^\beta \theta^{A\nu} \nabla_\beta U_{c\nu}) \end{aligned} \quad (9.19)$$

which we rewrite symbolically as:

$$([\mathcal{D}iv, \mathcal{L}_O]U)_c = \overline{E}_{\nu c}^\sigma U_\sigma^\nu + \sum_{A=1,2} \overline{\mathcal{H}}^{A\beta\nu} \nabla_\beta U_{c\nu} \quad (9.20)$$

Consequently, exploiting equation 9.3, if $N > 1$, the following relations holds with f a scalar function:

$$[\nabla^N, \frac{\mathcal{D}}{\partial\nu}]f = \sum_{k=0}^{N-1} \binom{N}{k} (\nabla^{N-1-k}\Omega\chi) \cdot \nabla^{k+1}f - \sum_{k=0}^{N-2} \binom{N}{k} (\nabla^{N-2-k}C) \cdot \nabla^{k+1}f, \quad (9.21)$$

the following one if V is an S -tangent vector field:

$$[\nabla^{N-1}, \frac{\mathcal{D}}{\partial\nu}]V = \sum_{k=0}^{N-2} \binom{N-1}{k} (\nabla^{N-2-k}\Omega\chi) \cdot \nabla^{k+1}V - \sum_{k=0}^{N-2} \binom{N}{k} (\nabla^{N-2-k}C) \cdot \nabla^kV. \quad (9.22)$$

in the same way we can calculate $[\mathcal{L}_O^{N-1}, \frac{\mathcal{D}}{\partial\nu}]V$

$$[\mathcal{L}_O^{N-1}, \frac{\mathcal{D}}{\partial\nu}]V = \sum_{k=0}^{N-2} \binom{N-1}{k} \sum_{k=0}^{N-2} \binom{N}{k} (\nabla^{N-2-k}E) \cdot \nabla^kV. \quad (9.23)$$

Lemma 9.2 $\mathcal{L}_O^{N-1}\psi$ satisfies the following equation along the outgoing cones,

$$\begin{aligned} \frac{\mathcal{D}}{\partial\nu}(\mathcal{L}_O^{N-1}\psi) + \frac{3}{2}\overline{\Omega tr\chi}(\hat{\mathcal{L}}_O^{N-1}\psi) &= -\Omega\hat{\chi} \cdot (\mathcal{L}_O^{N-1}\psi) - 2\hat{\chi}\mathcal{L}_O^{N-1}(\nabla\hat{\chi}) - tr\chi\mathcal{L}_O^{N-1}\beta + \\ &+ \frac{3}{2}(\overline{\Omega tr\chi} - \Omega tr\chi)(\mathcal{L}_O^{N-1}\psi) + \{(good)_1\} \end{aligned} \quad (9.24)$$

$$\begin{aligned} \{(good)_1\} &= 2 \sum_{k=1}^{N-1} \binom{N-1}{k} \mathcal{L}_O^{N-1-k}(\nabla\hat{\chi})\mathcal{L}_O^k\hat{\chi} \\ &- \frac{3}{2} \sum_{q=1}^{N-1} \binom{N-1}{q} (\mathcal{L}_O^q\Omega^2U)(\mathcal{L}_O^{N-1-q}\psi) + \sum_{k=0}^{N-1} \binom{N-1}{k} \mathcal{L}_O^{N-1-k}(|\hat{\chi}|^2)\mathcal{L}_O^k\eta \\ &+ \sum_{q=1}^{N-1} \binom{N-1}{q} (\mathcal{L}_O^q(\Omega\hat{\chi}))(\mathcal{L}_O^{N-1-q}\psi) + \sum_{q=0}^{N-2} \binom{N-1}{q} (\mathcal{L}_O^{N-2-q}E)(\mathcal{L}_O^q\psi) \\ &- \sum_{q=0}^{N-1} \binom{N-1}{q} (\mathcal{L}_O^q(\Omega U))(\mathcal{L}_O^{N-1-q}\hat{\chi}\eta) + \sum_{q=1}^{N-1} \binom{N-1}{q} (\mathcal{L}_O^{N-2-q}\beta)(\mathcal{L}_O^q tr\chi) \end{aligned} \quad (9.25)$$

Moreover the following inequality holds:

$$-\frac{\partial}{\partial\nu}|r^{(3-\frac{2}{p})}\mathcal{L}_O^{N-1}\psi|_{p,S} \leq |r^{(3-\frac{2}{p})}\mathcal{L}|_{p,S}. \quad (9.26)$$

where⁷³

⁷³In fact we have two inequalities

$$-|r^{(3-\frac{2}{p})}\mathcal{L}|_{p,S} \leq \frac{\partial}{\partial\nu}|r^{(3-\frac{2}{p})}\nabla^{N-1}\psi|_{p,S} \leq |r^{(3-\frac{2}{p})}\mathcal{L}|_{p,S}. \quad (9.27)$$

One is used when integrating along the outgoing cones the other ones for the incoming one.

$$\begin{aligned}
|\mathcal{L}| &= [|\Omega \hat{\chi} \cdot (\mathcal{L}_O^{N-1} \psi)| + |2\hat{\chi} \cdot (\mathcal{L}_O^{N-1} \nabla \hat{\chi})| + |\text{tr} \chi| |\mathcal{L}_O^{N-1} \beta| + |\{(good)_1\}|] \\
&+ \left[\frac{(3 - \frac{2}{p})}{2} |\Omega \text{tr} \chi - \overline{\Omega \text{tr} \chi}| |\mathcal{L}_O^{N-1} \psi| \right] \quad .
\end{aligned} \tag{9.28}$$

Proof: See Appendix 14.⁷⁴

To obtain from the differential inequality 14.56 an estimate for $\mathcal{L}_O^{N-1} \psi$ we need first to control the term $(\mathcal{L}_O^{N-1} \nabla \hat{\chi})$ present in \mathcal{L} , which has the maximal order of derivatives.⁷⁵ For it we use the Hodge equation

$$\text{div} \hat{\chi} = \frac{1}{2} \nabla \text{tr} \chi - \beta - \zeta \cdot \hat{\chi} + \frac{1}{2} \zeta \text{tr} \chi \equiv F^{(1)} \tag{9.29}$$

from the Hodge system 9.29 it is possible to prove the following lemma,

Lemma 9.3

$$|\mathcal{L}_O^{N-1} \nabla \hat{\chi}|_{p,S} \leq c (|\Omega|_{\infty,S} |\mathcal{L}_O^{N-1} \psi|_{p,S} + |\mathcal{L}_O^{N-1} \beta|_{p,S} + |\{(good)_2\}|_{p,S} + |\{(good)_4\}|_{p,S}) \tag{9.30}$$

with

$$\begin{aligned}
\{(good)_2\} &= [\text{div}, \mathcal{L}_O^{N-1}] \hat{\chi} \\
&= \sum_{q=0}^{N-2} \left(\binom{N-1}{q} (\mathcal{L}_O^{N-2-q} \overline{E})(\mathcal{L}_O^q \psi) + \binom{N-1}{q} (\mathcal{L}_O^{N-2-q} \overline{\mathcal{H}})(\mathcal{L}_O^q \nabla \psi) \right)
\end{aligned} \tag{9.31}$$

$$|\{(good)_4\}|_{p,S} = \left(\sum_{k=1}^{N-1} \binom{N-1}{k} |\mathcal{L}_O^k \Omega|_{\infty} |\mathcal{L}_O^{N-1-k} \hat{\chi}|_{p,S} + |\{(good)_3\}|_{p,S} \right) \tag{9.32}$$

$$\begin{aligned}
\{(good)_3\} &= -\mathcal{L}_O^{N-1} (\zeta \cdot \hat{\chi}) + [\text{div}, \mathcal{L}_O^{N-1}] \hat{\chi} \\
&= \mathcal{L}_O^{N-1} (\zeta \cdot \hat{\chi}) + \sum_{q=0}^{N-2} \left(\binom{N-1}{q} (\mathcal{L}_O^{N-2-q} \overline{E})(\mathcal{L}_O^q \psi) + \binom{N-1}{q} (\mathcal{L}_O^{N-2-q} \overline{\mathcal{H}})(\mathcal{L}_O^q \nabla \psi) \right)
\end{aligned} \tag{9.33}$$

⁷⁴Trying to use the analogous transport equation to estimate $|r^{N+2-\frac{2}{p}} \nabla^N U|_{p,S}$ does not work as the term $-\frac{U}{2} \sum_{k=1}^N \binom{N}{k} (\nabla^k \Omega^2)(\nabla^{N-k} U)$ gives rise, when integrated, to logarithmic divergent contributions.

⁷⁵Recall that ψ already contain a ∇ derivative.

The recursive assumption for $\mathcal{L}_O^{N-1}\psi$ is proved in the following theorem whose detailed proof is in the appendix to Section 9, Appendix 14, it is a prototype of all the remaining estimates of the connection coefficients angular derivatives.

Theorem 9.1 *Assuming the following norm estimate for the null Riemann component β , with $C^{(1)} = O(\varepsilon)$ for $J \leq 7$, $C^{(1)} = O(1)$ otherwise, $C^{(1)} < C_0$.⁷⁶*

$$|r^{2-\frac{2}{p}}\mathcal{L}_O^{N-1}\beta|_{p,S}(\lambda, \nu) \leq C^{(1)}e^{(N-2)\delta}e^{(N-2)\Gamma(\lambda)}\frac{N!}{N^\alpha}\frac{1}{\rho_{0,1}^N}, \quad (9.34)$$

assuming the norm estimates 3.31 for the initial data connection coefficients with $J \leq N$ and the norm estimates 3.29 for the connection coefficients with $J < N$, then the following inequality holds on \mathcal{K} ,

$$|r^{3-\frac{2}{p}}\mathcal{L}_O^{N-1}\psi|_{p,S}(\lambda, \nu) \leq C_0e^{(N-2)\delta}e^{(N-2)\Gamma(\lambda)}\frac{N!}{N^\alpha}\frac{1}{\rho_{0,1}^N}, \quad (9.35)$$

Proof: See Appendix 14.

Remark:

We are left with proving assumption 9.34 of Theorem 9.1. This is a particular case of the more general Theorem 4.1 which gives the appropriate estimates for the Riemann null components proved in detail in sections 10 and 15.

9.1.2 The control of $\mathcal{L}_O^N\eta$

The strategy to control the $|\cdot|_{p,S}$ norms of $\mathcal{L}_O^N\eta$ is based on the same approach described for the estimates of $\mathcal{L}_O^{N-1}\psi$. There are, nevertheless, some differences we are going to point out. First of all we cannot use in a straightforward way the transport equation for η ,

$$\mathbf{D}_4\eta + \frac{\text{tr}\chi}{2}\eta + \hat{\chi} \cdot \eta - \chi \cdot \underline{\eta} + \beta = 0 \quad (9.36)$$

as it implies a loss of derivatives (the Riemann coefficient β depends on second derivatives of the metric while η only on first ones). The idea, developed in [Ch-Kl] and extended in [Kl-Ni2], is to define the scalar quantity $\tilde{\mu}$

$$\tilde{\mu} := -\text{div}\eta + \frac{1}{2}(\chi\underline{\chi} - \overline{\chi\underline{\chi}}) - (\rho - \overline{\rho}) \quad (9.37)$$

and observe that, using the structure equations, $\tilde{\mu}$ satisfies the following transport equation,

$$\frac{\mathcal{D}}{\partial\nu}\tilde{\mu} + (\Omega\text{tr}\chi)\tilde{\mu} = (\Omega\tilde{F} - \overline{\Omega\tilde{F}}) + (\Omega\tilde{H} - \overline{\Omega\tilde{H}}) \quad (9.38)$$

⁷⁶It is important that the constant $C^{(1)}$, is smaller than all the constants estimating the connection coefficients, this is because of the fact that the null Riemann components appears in the structure equations not associated with other terms.

where

$$\begin{aligned}\tilde{F} &= \hat{\chi} \cdot (\nabla \hat{\otimes} \eta) + \Omega(\eta - \underline{\eta}) \nabla (\Omega^{-1} \text{tr} \chi) + \frac{1}{2} \text{tr} \chi (|\eta|^2 - |\underline{\eta}|^2) - \frac{1}{2} (\text{tr} \chi |\hat{\chi}|^2 + \text{tr} \chi (\hat{\chi} \cdot \underline{\hat{\chi}})) + 2\eta \cdot \hat{\chi} \cdot \underline{\eta} \\ \tilde{H} &= \left(-\frac{\text{tr} \chi}{2} \overline{\underline{\chi} \cdot \underline{\chi}} - 2\eta \cdot \beta + \text{tr} \chi (\rho + \bar{\rho}) \right) .\end{aligned}\tag{9.39}$$

A long but obvious computation, analogous to the one for $\mathcal{L}_O^{N-1} \psi$, proves the following lemma,

Lemma 9.4 $\mathcal{L}_O^{N-1} \tilde{\mu}$ satisfies the following transport equation along the outgoing cones, for $N > 1$,

$$\begin{aligned}\frac{\mathcal{D}(\mathcal{L}_O^{N-1} \tilde{\mu})}{\partial \nu} + \Omega \text{tr} \chi \cdot (\mathcal{L}_O^{N-1} \tilde{\mu}) - \hat{\chi} \cdot \mathcal{L}_O^{N-1} (\nabla \hat{\otimes} \eta) \\ + 2\Omega \eta \cdot \mathcal{L}_O^{N-1} \beta - \Omega \text{tr} \chi \mathcal{L}_O^{N-1} \rho + \left[\{(\widetilde{good})_1\} + \{(\widetilde{good})_2\} \right] = 0 ,\end{aligned}\tag{9.40}$$

where

$$\begin{aligned}\{(\widetilde{good})_1\} &= - \sum_{q=0}^{N-2} \binom{N-1}{q} (\mathcal{L}_O^{N-2-q} E) (\mathcal{L}_O^q \tilde{\mu}) \\ \{(\widetilde{good})_2\} &= - \left(- \sum_{k=1}^{N-1} \binom{N-1}{k} \mathcal{L}_O^k (\Omega \text{tr} \chi) (\mathcal{L}_O^{N-1-k} \tilde{\mu}) + \mathcal{L}_O^{N-1} \left[(\Omega \tilde{F} - \overline{\Omega \tilde{F}}) + (\Omega \tilde{H} - \overline{\Omega \tilde{H}}) \right] \right. \\ &\quad \left. - \hat{\chi} \cdot \mathcal{L}_O^{N-1} (\nabla \hat{\otimes} \eta) + 2\Omega \eta \cdot \mathcal{L}_O^{N-1} \beta - \Omega \text{tr} \chi \mathcal{L}_O^{N-1} \rho \right) .\end{aligned}\tag{9.41}$$

and

$$\{(\widetilde{good})_{2+1}\} = \{(\widetilde{good})_1\} + \{(\widetilde{good})_2\}\tag{9.42}$$

Moreover the following inequality⁷⁷ holds

$$- \frac{\partial |r^{(2-\frac{2}{p})} \mathcal{L}_O^{N-1} \tilde{\mu}|_{p,S}}{\partial \nu} \leq |r^{(2-\frac{2}{p})} \tilde{L}|_{p,S} ,\tag{9.43}$$

where

$$\begin{aligned}|\tilde{L}| \leq & \left[|\hat{\chi} \cdot (\mathcal{L}_O^{N-1} \tilde{\mu})| + |\hat{\chi} \cdot (\mathcal{L}_O^{N-1} \nabla \hat{\otimes} \eta)| + |\eta| |\mathcal{L}_O^{N-1} \beta| + |\text{tr} \chi| |\mathcal{L}_O^{N-1} \rho| + |\{(\widetilde{good})_{1+2}\}| \right] \\ & + [|\Omega \text{tr} \chi - \overline{\Omega \text{tr} \chi}| |\mathcal{L}_O^{N-1} \tilde{\mu}|] .\end{aligned}\tag{9.44}$$

Proof: The proof has the same structure as the proof of Lemma 9.2 and we do not report it here. In fact observing that η satisfies the following Hodge system

$$\begin{aligned}\text{div} \eta &= -\tilde{\mu} + \frac{1}{2} (\chi \underline{\chi} - \overline{\chi \underline{\chi}}) - (\rho - \bar{\rho}) \equiv F^{(0)} \\ \text{curl} \eta &= \sigma - \frac{1}{2} \underline{\hat{\chi}} \wedge \hat{\chi} \equiv G^{(0)} .\end{aligned}\tag{9.45}$$

⁷⁷Analogous to inequality 9.26.

and examining equations 9.40, 9.45 it is immediate to recognize to that:

- i) $\tilde{\mu}$ plays the role of ψ ⁷⁸ and the terms in $\{(\widetilde{good})_1\} + \{(\widetilde{good})_2\}$ contain only (angular) derivatives of the connection coefficients lower than N or terms of highest derivative order which have already been controlled in the previous step. In this sense all these terms are “good terms”.
- ii) As in equation 9.26 for $\mathcal{L}_O^{N-1}\psi$ the terms $\mathcal{L}_O^{N-1}\nabla\hat{\chi}$ are controlled deriving the Hodge system 9.29, here the term $\mathcal{L}_O^{N-1}(\nabla\hat{\otimes}\eta)$ is controlled in the same way deriving the Hodge system 9.45. The result of this estimate, we do not report here, is in Lemma 9.5, the analogous of Lemma 9.3.
- iii) Finally, as in the estimate for $\mathcal{L}_O^{N-1}\psi$, some Riemann null components with the highest order of derivatives ($N-1$ for the Riemann components) appear in the transport equation 9.40 and in the Hodge system 9.45. Again these terms have to be estimated using Theorem 4.1.

Lemma 9.5 *Assuming that η satisfies the Hodge system 9.45 then $\mathcal{L}_O^{N-1}(\nabla\hat{\otimes}\eta)$ satisfies the following inequality,*

$$|\nabla\mathcal{L}_O^{N-1}\eta|_{p,S} \leq c \left(|\mathcal{L}_O^{N-1}\tilde{\mu}|_{p,S} + |\mathcal{L}_O^{N-1}\rho|_{p,S} + |\{(\widetilde{good})_3\}|_{p,S} \right) \quad (9.46)$$

With c a suitable constant and

$$\begin{aligned} \{(\widetilde{good})_3\} &= -\mathcal{L}_O^{N-1}\frac{1}{2}(\chi\underline{\chi} - \overline{\chi}\overline{\underline{\chi}}) + \mathcal{L}_O^{N-2}\rho + [\text{div}, \mathcal{L}_O^{N-1}]\eta \\ &= -\mathcal{L}_O^{N-1}\left(\frac{1}{2}(\chi\underline{\chi} - \overline{\chi}\overline{\underline{\chi}}) + \mathcal{L}_O^{N-2}\rho + \sum_{q=0}^{N-2} \binom{N-1}{q} (\mathcal{L}_O^{N-2-q}\overline{E})(\mathcal{L}_O^q\eta) \right. \\ &\quad \left. + \binom{N-1}{q} (\mathcal{L}_O^{N-2-q}\overline{\mathcal{H}})(\mathcal{L}_O^q\nabla\eta) \right) \end{aligned} \quad (9.47)$$

consequently

$$|\mathcal{L}_O^{N-1}\nabla\eta|_{p,S} \leq c \left(|\mathcal{L}_O^{N-1}\tilde{\mu}|_{p,S} + |\mathcal{L}_O^{N-1}\rho|_{p,S} + |\{(\widetilde{good})_3\}|_{p,S} + |\{(\widetilde{good})_4\}|_{p,S} \right) \quad (9.48)$$

With $|\{(\widetilde{good})_4\}|_{p,S} = [\nabla, \mathcal{L}_O^{N-1}]\eta$

hence, the same inequality holds for $\mathcal{L}_O^{N-1}\nabla\hat{\otimes}\eta$

$$|\mathcal{L}_O^{N-1}(\nabla\hat{\otimes}\eta)|_{p,S} \leq c \left(|\mathcal{L}_O^{N-1}\tilde{\mu}|_{p,S} + |\mathcal{L}_O^{N-1}\rho|_{p,S} + |\{(\widetilde{good})_3\}|_{p,S} + |\{(\widetilde{good})_4\}|_{p,S} \right) \quad (9.49)$$

⁷⁸There are $N-1$ derivatives in 9.40 as, in $\tilde{\mu}$, η is already derived once.

Retracing the steps performed for Ψ we can hence obtain the estimate for $\mathcal{L}_O^{N-1}\tilde{\mu}$, stated in the following theorem.

Theorem 9.2 *Assuming the norm estimates for the null Riemann components of Theorem 4.1, assuming the norm estimates 3.31 for the initial data connection coefficients with $J \leq N$ and the norm estimates 3.29 for the connection coefficients with $J < N$, assuming that on the “last slice” $\mathcal{L}_O^{N-1}\tilde{\mu}$ satisfy the following estimate*

$$|r^{(2-\frac{2}{p})}\mathcal{L}_O^{N-1}\tilde{\mu}|_{p,S}(\lambda, \nu_*) \leq C_6^{(*)} \left(\frac{N!}{N^\alpha} e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \frac{1}{\rho_{0,1}^N} \right), \quad (9.50)$$

then the following inequality holds with $C_6^{(*)} < C_6$,

$$|r^{(2-\frac{2}{p})}\mathcal{L}_O^{N-1}\tilde{\mu}|_{p,S}(\lambda, \nu) \leq \tilde{C}_6 e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} \quad (9.51)$$

And by definition of $\tilde{\mu}$ we obtain easily the inequality for η ,

$$|r^{(2-\frac{2}{p})}\mathcal{L}_O^{N-1}\nabla\eta|_{p,S}(\lambda, \nu) \leq C_6 e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} \quad (9.52)$$

With $\tilde{C}_6 < C_6$.

9.1.3 The control of $\mathcal{L}_O^N \mathbf{D}_3 \log \Omega$

To control $\mathcal{L}_O^N \underline{\omega} = -2^{-1} \mathcal{L}_O^N \mathbf{D}_3 \log \Omega$ we cannot use the transport equation for $\underline{\omega}$. This is due to the fact that in its transport equation

$$\mathbf{D}_4 \underline{\omega} - 2\omega \underline{\omega} - \zeta \cdot \nabla \log \Omega - \frac{3}{2} |\zeta|^2 + \frac{1}{2} |\nabla \log \Omega|^2 - \frac{1}{2} \rho = 0$$

there is a loss of derivatives, namely this equation depends on the Riemann component ρ . This problem is analogous to the one we faced in the control of $\mathcal{L}_O^N \eta$ and the way out⁷⁹ is exactly of the same type as the one used for $\mathcal{L}_O^N \eta$. Instead of $\tilde{\mu}$ we introduce the following quantity,

$$\underline{\tilde{\omega}} \equiv \text{div } \hat{V} \quad \text{where} \quad \hat{V} = (\tilde{V} - \Omega \underline{\beta}), \quad \text{and} \quad \tilde{V} = \nabla \Omega \mathbf{D}_3 \log \Omega = -2 \nabla \Omega \underline{\omega}. \quad (9.53)$$

With standard computations we obtain,

⁷⁹This is an adaptation of a strategy first used by D.Christodoulou and S.Klainerman in [Ch-Kl].

Lemma 9.6 *The structure equation for $\underline{\tilde{\omega}}$ is*

$$\begin{aligned} \frac{\partial \underline{\tilde{\omega}}}{\partial \nu} + \Omega \operatorname{tr} \chi \underline{\tilde{\omega}} + 4\Omega \omega \underline{\tilde{\omega}} &= -2\Omega \hat{\chi} \cdot \nabla \hat{V} \\ + \Omega \left[-2\hat{\chi} \cdot \nabla \Omega \underline{\beta} - 4\omega \operatorname{div}(\Omega \underline{\beta}) + \Omega(\nabla \log \Omega) \cdot \nabla \rho + \frac{1}{2} \operatorname{tr} \chi \Omega \operatorname{div} \underline{\beta} + \hat{\chi} \cdot \Omega \nabla \beta + (\nabla \Omega)(-\nabla \rho + {}^* \nabla \sigma) \right] \\ + \{\mathcal{F}\} . \end{aligned} \quad (9.54)$$

where $\{\mathcal{F}\}$ collects terms with less derivatives whose explicit expression is in Appendix 14. Moreover the transport equation for $\mathcal{L}_O^{N-2} \underline{\tilde{\omega}}$ is:

$$\begin{aligned} \frac{\partial}{\partial \nu} (\mathcal{L}_O^{N-2} \underline{\tilde{\omega}}) + (\Omega \operatorname{tr} \chi + 4\Omega \omega) (\mathcal{L}_O^{N-2} \underline{\tilde{\omega}}) + (\Omega \chi) \cdot (\mathcal{L}_O^{N-2} \underline{\tilde{\omega}}) \\ = -2\Omega \hat{\chi} \mathcal{L}_O^{N-2} \nabla \hat{V} + \Omega \left[-2\Omega \hat{\chi} \cdot \mathcal{L}_O^{N-1} \underline{\beta} - 4\Omega \omega \mathcal{L}_O^{N-2} \operatorname{div} \underline{\beta} + \Omega(\nabla \log \Omega) \cdot \mathcal{L}_O^{N-1} \rho \right. \\ \left. + \frac{1}{2} \operatorname{tr} \chi \Omega \mathcal{L}_O^{N-2} \operatorname{div} \underline{\beta} + \hat{\chi} \cdot \Omega \mathcal{L}_O^{N-1} \beta + (\nabla \Omega)(-\mathcal{L}_O^{N-1} \rho + \mathcal{L}_O^{N-2} {}^* \nabla \sigma) \right] \\ + \{\mathcal{L}\} + \{\mathcal{G}\} + \{\tilde{\mathcal{F}}\} , \end{aligned} \quad (9.55)$$

where $\{\mathcal{L}\}$, $\{\mathcal{G}\}$, $\{\tilde{\mathcal{F}}\}$, whose explicit expressions are given in Appendix 14, collect all the terms with lower order derivatives which can be estimated using the inductive assumptions.

Remark:

Notice that in the terms $\{\mathcal{L}\} + \{\mathcal{G}\} + \{\tilde{\mathcal{F}}\}$ the Lie derivatives of ζ appear, up to order $N-1$, which has been not explicitly estimated, this is not a problem as remember that $\zeta = \eta + \nabla \log \Omega$ and these quantities are already at our disposal by the inductive assumptions.

No loss of derivatives is present in equation 9.54. In fact $\underline{\tilde{\omega}}$ depends on three derivatives of the metric components as all the terms explicitly written on the r.h.s. while $\{\mathcal{F}\}$ depends on lower derivatives.

Equation 9.55 plays the same role of the transport equation for $\mathcal{L}_O^{N-1} \tilde{\mu}$ or the one for $\mathcal{L}_O^{N-1} \psi$; It depends on \mathcal{L}_O^{N-1} derivatives of Riemann components which have to be estimated using the hyperbolicity, see Theorem 4.1, and on $\mathcal{L}_O^{N-2} \nabla \hat{V}$; this last term is the analogous of the term $\nabla^{N-1}(\nabla \hat{\otimes} \eta)$ present in the equation 9.40 or to the term $\mathcal{L}_O^{N-1} \nabla \hat{\chi}$ present in equation 9.25 for $\mathcal{L}_O^{N-1} \psi$. To control the $\mathcal{L}_O^{N-1} \hat{V}$ term in terms of $\underline{\tilde{\omega}}$ and from it to control the norms of $\mathcal{L}_O^{N-2} \underline{\tilde{\omega}}$ applying Gronwall's Lemma, we have to use the Hodge system,

$$\operatorname{div} \hat{V} = \underline{\tilde{\omega}} , \quad (9.56)$$

which is equivalent, but easier to treat, to the Hodge system 9.29 and to the Hodge system 9.45 used to control the η derivatives from the $\tilde{\mu}$ estimates. Starting from 9.56 the Hodge system to consider is provided in the following lemma, whose simple proof we do not report here,

Lemma 9.7 *From the Hodge system 9.56 it follows that $\nabla^{N-1}\hat{V}$ satisfies the following estimates*

$$|\mathcal{L}_O^{N-2}\nabla\hat{V}|_{p,S} \leq |[div, \mathcal{L}_O^{N-2}]\hat{V}|_{p,S} + |\widetilde{\{good_3\}}|_{p,S} + |\tilde{\omega}|_{p,S} \quad (9.57)$$

with

$$|\widetilde{\{good_3\}}|_{p,S} = |[div, \mathcal{L}_O^{N-2}]\hat{V}|_{p,S} \quad (9.58)$$

Finally once we control the norms of $\mathcal{L}_O^{N-2}\tilde{\omega}$, as from Theorem 4.1 we control $\mathcal{L}_O^{N-2}\tilde{\omega}$ it follows that we control $\mathcal{L}_O^{N-2}\omega$ and from it we control the norm of $\nabla^N\omega$ which is the final step required to prove the following theorem, analogous to Theorem 9.2,

Theorem 9.3 *Assuming the results of Theorem 4.1 for the null Riemann components, the inductive assumptions for the connection coefficients up to $N-1$ derivatives and the result of previous Theorems 9.1 and 9.2, assuming that on the “last slice” $\mathcal{L}_O^{N-2}\tilde{\omega}$ satisfy the following estimate with $\underline{C}_2 > \tilde{C}_2 > \tilde{C}_2^{(*)}$,*

$$|r^{3-\frac{2}{p}}|\lambda|\mathcal{L}_O^{N-2}\tilde{\omega}|_{p,S}(\lambda, \nu_*) \leq \tilde{C}_2^{(*)} e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \frac{N!}{N^\alpha \rho_{0,1}^N}, \quad (9.59)$$

then the following inequality holds

$$|r^{3-\frac{2}{p}}|\lambda|\mathcal{L}_O^{N-2}\tilde{\omega}|_{p,S}(\lambda, \nu) \leq \tilde{C}_2 e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N}, \quad (9.60)$$

$$|r^{2-\frac{2}{p}}|\lambda|\mathcal{L}_O^N\omega|_{p,S}(\lambda, \nu) \leq \underline{C}_2 e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N}. \quad (9.61)$$

9.1.4 The control of the \mathcal{L}_O^N derivatives of the underlined connection coefficients

To complete the proof of Theorem 3.1 we have to obtain the norm estimates for the \mathcal{L}_O^N angular derivatives of the underlined connection coefficients, $\underline{\chi}, \underline{\eta}, \omega$ ⁸⁰ whose transport equations are along the incoming cones. The procedure is analogous to the one we discussed for $\chi, \eta, \underline{\omega}$ and we look in detail only to the estimates for $\mathcal{L}_O^{N-1}\underline{\omega}$ to point out the main differences, the most important being that in this case the estimates are done “from below”, namely the integration starts from the initial data on C_0 . As in the $\nabla^N\chi$ case, we start proving the norm estimates for $\nabla^{N-1}\underline{\omega}$ where $\underline{\omega} = \nabla\Omega^{-1}\text{tr}\underline{\chi} + \Omega^{-1}\text{tr}\underline{\chi}\underline{\eta}$.

⁸⁰To follow previous notation, see [Kl-Ni2] we denoted with ω a connection coefficient whose transport equation is along the incoming cones.

To obtain the transport equation for $\mathcal{L}_O^{N-1}\underline{\Psi}$ we apply \mathcal{L}_O^{N-1} to the structure equation⁸¹

$$\frac{\mathcal{D}}{\partial\lambda}\underline{\Psi} + \frac{3}{2}\Omega\text{tr}\underline{\chi}\underline{\Psi} = -\nabla|\underline{\hat{\chi}}|^2 - \underline{\eta}|\underline{\hat{\chi}}|^2 - \Omega\underline{\hat{\chi}} \cdot \underline{\Psi} + \text{tr}\underline{\chi}(\underline{\hat{\chi}} \cdot \underline{\eta}) - \text{tr}\underline{\chi}\underline{\beta}. \quad (9.62)$$

The result we obtain is, therefore, provided in the following lemma, whose proof is identical to the proof of Lemma 9.2 with the obvious substitutions,

Lemma 9.8 *Denoting $\underline{\Psi} = \nabla\underline{U} + \underline{U}\underline{\eta}$ where $\underline{U} = \Omega^{-1}\text{tr}\underline{\chi}$, $\mathcal{L}_O^{N-1}\underline{\Psi}$ satisfies the following transport equation along the incoming cones,*

$$\begin{aligned} \frac{\mathcal{D}}{\partial\lambda}(\mathcal{L}_O^{N-1}\underline{\Psi}) + \frac{3}{2}\overline{\Omega\text{tr}\underline{\chi}}(\hat{\mathcal{L}}_O^{N-1}\underline{\Psi}) &= -\Omega\underline{\hat{\chi}} \cdot (\hat{\mathcal{L}}_O^{N-1}\underline{\Psi}) - 2\underline{\hat{\chi}}\hat{\mathcal{L}}_O^{N-1}(\nabla\underline{\hat{\chi}}) - \text{tr}\underline{\chi}\mathcal{L}_O^{N-1}\underline{\beta} + \\ &+ \frac{3}{2}(\overline{\Omega\text{tr}\underline{\chi}} - \Omega\text{tr}\underline{\chi})(\hat{\mathcal{L}}_O^{N-1}\underline{\Psi}) + \{(\underline{good})_1\} \end{aligned} \quad (9.63)$$

With

$$\begin{aligned} \{(\underline{good})_1\} &= 2 \sum_{k=1}^{N-1} \binom{N-1}{k} \mathcal{L}_O^{N-1-k}(\nabla\underline{\hat{\chi}})\mathcal{L}_O^k\underline{\hat{\chi}} \\ &- \frac{3}{2} \sum_{q=1}^{N-1} \binom{N-1}{q} (\hat{\mathcal{L}}_O^q\Omega^2\underline{U})(\hat{\mathcal{L}}_O^{N-1-q}\underline{\Psi}) + \sum_{k=0}^{N-1} \binom{N-1}{k} \mathcal{L}_O^{N-1-k}(|\underline{\hat{\chi}}|^2)\mathcal{L}_O^k\underline{\eta} \\ &+ \sum_{q=1}^{N-1} \binom{N-1}{q} (\mathcal{L}_O^q(\Omega\underline{\hat{\chi}}))(\mathcal{L}_O^{N-1-q}\underline{\Psi}) + \sum_{q=0}^{N-2} \binom{N-1}{q} (\mathcal{L}_O^{N-2-q}\underline{E})(\mathcal{L}_O^q\underline{\Psi}) \\ &- \sum_{q=0}^{N-1} \binom{N-1}{q} (\mathcal{L}_O^q(\Omega\underline{U}))(\mathcal{L}_O^{N-1-q}\underline{\hat{\chi}}\underline{\eta}) + \sum_{q=1}^{N-1} \binom{N-1}{q} (\mathcal{L}_O^{N-2-q}\underline{\beta})(\mathcal{L}_O^q\text{tr}\underline{\chi}) \end{aligned}$$

and with \underline{E} defined in equation 9.17.

Moreover the following inequality holds:

$$\frac{\partial}{\partial\lambda}|r^{(3-\frac{2}{p})}\mathcal{L}_O^{N-1}\underline{\Psi}|_{p,S} \leq |r^{(3-\frac{2}{p})}\underline{L}|_{p,S}, \quad (9.64)$$

$$\begin{aligned} |\underline{L}| &= [|\Omega\underline{\hat{\chi}} \cdot (\mathcal{L}_O^{N-1}\underline{\Psi})| + |2\underline{\hat{\chi}} \cdot (\mathcal{L}_O^{N-1}\nabla\underline{\hat{\chi}})| + |\text{tr}\underline{\chi}||\mathcal{L}_O^{N-1}\underline{\beta}| + |\{(\underline{good})_1\}|] \\ &+ \left[\frac{(3-\frac{2}{p})}{2} |\Omega\text{tr}\underline{\chi} - \overline{\Omega\text{tr}\underline{\chi}}| |\mathcal{L}_O^{N-1}\underline{\Psi}| \right]. \end{aligned} \quad (9.65)$$

⁸¹Observe that equation 9.62 has to be considered a tensorial equation. As we are interested to the transport equation for its $L_{p,S}$ norm, we do not need to express e_3 as $\frac{1}{\Omega}(\frac{\partial}{\partial\lambda} + X^a\frac{\partial}{\partial\omega^a})$, see (I); in fact in this case we can choose on the $S(\lambda, \nu)$ surfaces an orthonormal frame Fermi transported along the incoming cones. In this way the transport equations along the incoming cones for these norms are exactly the same as those previously obtained for the not underlined coefficients just substituting ν with λ and interchanging the underlined with the not underlined connection coefficients and Riemann components.

Exactly as in the estimate of $\mathcal{L}_O^{N-1}\underline{\psi}$, to obtain from the differential inequality 9.64 an estimate for $\mathcal{L}_O^{N-1}\underline{\psi}$ we need first to control the term $\mathcal{L}_O\nabla\hat{\chi}$ present in \underline{L} , which has the same N order of derivatives. As before we use repeatedly the equations

$$\mathfrak{div}\hat{\chi} = \frac{1}{2}\nabla\text{tr}\chi - \underline{\beta} + \zeta \cdot \hat{\chi} - \frac{1}{2}\zeta\text{tr}\chi \equiv \underline{F}^{(1)} \quad (9.66)$$

and proceeding exactly as in Lemma 9.3 we prove the following lemma,

Lemma 9.9 *From equation 9.66 we obtain the following estimates*

$$|\mathcal{L}_O^{N-1}\nabla\hat{\chi}|_{p,S} \leq c(|\Omega|_{\infty,S}|\mathcal{L}_O^{N-1}\underline{\psi}|_{p,S} + |\mathcal{L}_O^{N-1}\underline{\beta}|_{p,S} + |\{(\underline{good})_2\}|_{p,S} + |\{(\underline{good})_4\}|_{p,S}) \quad (9.67)$$

with, c a suitable constant and

$$\begin{aligned} \{(\underline{good})_2\} &= [\mathfrak{div}, \mathcal{L}_O^{N-1}]\hat{\chi} \\ &= \sum_{q=0}^{N-2} \left(\binom{N-1}{q} (\mathcal{L}_O^{N-2-q}\overline{E})(\mathcal{L}_O^q\underline{\psi}) + \binom{N-1}{q} (\mathcal{L}_O^{N-2-q}\overline{\mathcal{H}})(\mathcal{L}_O^q\nabla\underline{\psi}) \right) \end{aligned} \quad (9.68)$$

$$|\{(\underline{good})_4\}|_{p,S} = \left(\sum_{k=1}^{N-1} \binom{N-1}{k} |\mathcal{L}_O^k\Omega|_{\infty} |\mathcal{L}_O^{N-1-k}\hat{\chi}|_{p,S} + |\{(\underline{good})_3\}|_{p,S} \right) \quad (9.69)$$

$$\begin{aligned} \{(\underline{good})_3\} &= -\mathcal{L}_O^{N-1}(\zeta \cdot \hat{\chi}) + [\mathfrak{div}, \mathcal{L}_O^{N-1}]\hat{\chi} \\ &= \mathcal{L}_O^{N-1}(\zeta \cdot \hat{\chi}) + \sum_{q=0}^{N-2} \left(\binom{N-1}{q} (\mathcal{L}_O^{N-2-q}\overline{E})(\mathcal{L}_O^q\underline{\psi}) + \binom{N-1}{q} (\mathcal{L}_O^{N-2-q}\overline{\mathcal{H}})(\mathcal{L}_O^q\nabla\underline{\psi}) \right) \end{aligned} \quad (9.70)$$

Using the estimate for $\mathcal{L}_O^{N-1}\underline{\beta}$ proved in Theorem 4.1, we prove the analogous of Theorem 9.1.

Theorem 9.4 *Assuming the following norm estimate for the null Riemann component $\underline{\beta}$,*

$$|r^{2-\frac{2}{p}}\mathcal{L}_O^{N-1}\underline{\beta}|_{p,S}(\lambda, \nu) \leq C^{(1)}e^{(N-2)\delta}e^{(N-2)\Gamma(\lambda)}\frac{N!}{N^\alpha}\frac{1}{\rho_{0,1}^N}, \quad (9.71)$$

assuming the norm estimates 3.31 for the initial data connection coefficients with $J \leq N$ and the norm estimates 3.29 for the connection coefficients with $J < N$, then the following inequality holds on \mathcal{K} ,

$$|r^{(N+2)-\frac{2}{p}} \mathcal{L}_O^{N-1} \underline{\Psi}|_{p,S}(\lambda, \nu) \leq \underline{C}_0 e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N}. \quad (9.72)$$

Proof: The proof is similar to the one of theorem 9.1 and we do not repeat it here.

With this theorem we can consider proved the estimates 3.26.

In order to prove the estimates 3.27 we state the following

Corollary 9.5 *Assuming the estimates 3.26 the following estimates hold on \mathcal{K} , with C a suitable constant.*

$$|r^{2-\frac{2}{p}} \mathcal{L}_O^N \{\mathcal{O}, \underline{\mathcal{O}}\}|_{p,S} \leq C e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} \quad (9.73)$$

Proof: See Appendix 14.

In order to obtain the \mathcal{L}_O^N estimates for the connection coefficients, we have to pass from the ∇^N estimates on the initial data to the \mathcal{L}_O^N ones, to do this we have to perform estimates of some auxiliary quantities, with the same structure of the connection coefficients, see equations 15.13 and 15.29. Viceversa, once we have the \mathcal{L}_O^N estimates for the connection coefficient in all the region, we have to recover the ∇^N , to do this we have to exploit the estimates for some other auxiliary quantities. We prove these estimates in the following section.

9.2 The $\mathbf{g}(e_A \nabla . e_B)$ and $\nabla_A O$ coefficients

In the proof of our result it follows that many different connection coefficients play some role, even if they do not enter in the Einstein equations. Their estimates are needed when we have to control the angular derivatives of the Riemann components, first of all, when, starting from the control of the ∇ we pass to the ∇_O derivatives of the initial data, see equation 15.13, we need the estimates of $\nabla_A O$, and of $\mathbf{g}(e_A, \nabla e_C)$ in order to do this we recall that the rotation vectors satisfy the equations, see [Kl-Ni2], section 4.6.

$$\frac{\mathcal{D}}{\partial \nu} {}^{(i)}O_b = \Omega \chi_{bc} {}^{(i)}O_c. \quad (9.74)$$

We can observe that these equations can be treated as the structure equations for the connection coefficients, hence with an analogous procedure to the one discussed previously we can obtain the estimates for $\nabla^N {}^{(i)}O$, on the initial data

and estimates for $\mathcal{L}_O^{(i)}O$ in the interior region. We do not repeat it here the proofs, finally we obtain the following theorems. ^{82 83}

Theorem 9.6 *Assuming the norm estimates for the null Riemann component of Theorem 4.1, assuming the norm estimates 3.31 for the initial data connection coefficients, the norm estimates 3.29 for the connection coefficients with $J < N$ and the following estimate for $^{(i)}O_b$, with $J < N$, c a suitable constant*

$$|r^{(J+1-\frac{2}{p})}\mathcal{L}_O^J{}^{(i)}O|_{p,S} \leq c \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right),$$

then the following inequalities hold on \mathcal{K} ,

$$|r^{N+1-\frac{2}{p}}\mathcal{L}_O^N{}^{(i)}O|_{p,S}(\lambda, \nu) \leq c \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)\delta}e^{(N-2)\Gamma(\lambda)}}{\rho_{0,1}^N} \right). \quad (9.75)$$

$$|r^{N+1-\frac{2}{p}}\mathcal{L}_O^{N-1}{}^{(i)}O|_{\infty,S}(\lambda, \nu) \leq c \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)\delta}e^{(N-2)\Gamma(\lambda)}}{\rho_{0,1}^N} \right). \quad (9.76)$$

Corollary 9.7 *Under the assumptions of Theorem 9.6 the following estimates hold on \mathcal{K} ,*

$$|r^{N+1-\frac{2}{p}}\nabla_O^N{}^{(i)}O|_{p,S}(\lambda, \nu) \leq c \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)\delta}e^{(N-2)\Gamma(\lambda)}}{\rho_{0,1}^N} \right). \quad (9.77)$$

$$|r^{N+1-\frac{2}{p}}\nabla_O^{N-1}{}^{(i)}O|_{\infty,S}(\lambda, \nu) \leq c \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)\delta}e^{(N-2)\Gamma(\lambda)}}{\rho_{0,1}^N} \right). \quad (9.78)$$

The proof is a repetition of lemma 4.2.

Conversely, once we have the estimates for the \mathcal{L}_O^N derivatives of the connection coefficients in all the region, in order to recover the ∇^N ones, we have to estimates the quantities $\nabla^J e_A$, $J < N$ and hence $\mathbf{g}(e_C, \nabla^J e_A)$, see equation 15.105, from the \mathcal{L}_O estimates. Let us proceed step by step: First let us prove

⁸²Recall that the sup norm of the J derivative is estimated in term of the $|\cdot|_{p,S}$ norm of the J and of the $J+1$ derivative. In fact the second line is

$$|r^{\phi(O)}\nabla^J\mathcal{O}|_{\infty,S} \leq C \left[\frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0}e^{(J-2)\Gamma_0(\lambda)}}{\rho_0^J} + \frac{(J+1)!}{(J+1)^\alpha} \frac{e^{(J-1)\delta_0}e^{(J-1)\Gamma_0(\lambda)}}{\rho_0^{(J+1)}} \right] \leq$$

$$cC \frac{(J+1)!}{(J+1)^\alpha} \frac{e^{(J-1)\delta_0}e^{(J-1)\Gamma_0(\lambda)}}{\rho_0^{(J+1)}} \text{ and we neglect } c \geq 1 + \frac{\rho_0}{(J+1)e^{\delta_0+\Gamma_0(\lambda)}}.$$

⁸³The estimate in the second line follows as (assuming $r=1$) although the O generators are at the metric level, nevertheless in the estimates of the angular derivatives of γ_{ab} and of X^a there is a loss of derivatives, see Section 5.2, which implies the same estimates as for the connection coefficients.

the estimates for $\mathbf{g}(e_C, \mathcal{L}_O^N e_A)$. Assuming e_A Fermi transported, $\mathcal{D}_4 e_A = 0$, it follows

$$\begin{aligned} \frac{\partial}{\partial \nu} \mathbf{g}(e_A, \nabla_B e_C) &= \mathbf{g}(e_A, \mathcal{D}_\nu \nabla_B e_C) = \mathbf{g}(e_A, [\mathcal{D}_\nu, \nabla_B] e_C) \\ &= e_B^\mu \mathbf{g}(e_A, [\mathcal{D}_\nu, \nabla_\mu] e_C) = e_B^\mu \mathbf{g}(e_A, (-\Omega \chi_\mu \cdot \nabla) e_C) = -\Omega \chi(e_B, e_D) \mathbf{g}(e_A, \nabla_D e_C) \\ &= -\frac{\Omega \text{tr} \chi}{2} \mathbf{g}(e_A, \nabla_B e_C) - \Omega \hat{\chi}_B \cdot \mathbf{g}(e_A, \nabla e_C) . \end{aligned} \quad (9.79)$$

Therefore we have the following transport equation

$$\frac{\mathcal{D}}{\partial \nu} \mathbf{g}(e_A, \nabla e_C) + \frac{\Omega \text{tr} \chi}{2} \mathbf{g}(e_A, \nabla e_C) = -\Omega \hat{\chi} \cdot \mathbf{g}(e_A, \nabla e_C) . \quad (9.80)$$

which we rewrite, recalling that $\mathbf{g}(e_A, \nabla e_C)$ is an S -tangent one form, we denote simply by $S_{AC}(\cdot) \equiv \mathbf{g}(e_A, \nabla e_C)$ or simpler by $S = S(\cdot)$,

$$\frac{\mathcal{D}}{\partial \nu} S + \frac{\Omega \text{tr} \chi}{2} S + \Omega \hat{\chi} \cdot S = 0 . \quad (9.81)$$

To obtain the estimate for $\mathcal{L}_O^N S$ we proceed as in the cases of the standard connection coefficient, discussed previously, and obtain the following results

Theorem 9.8 *Assuming the norm estimates for the null Riemann component of Theorem 4.1, assuming the norm estimates 3.31 for the initial data connection coefficients, the norm estimates 3.29 for the connection coefficients with $J < N$ and the following estimate for $\mathcal{L}_O^J S$, with $J < N$, c a suitable constant,*

$$|r^{(1-\frac{2}{p})} \mathcal{L}_O^J \mathbf{g}(e_A, \nabla e_C)|_{p,S} \leq c \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right) , \quad (9.82)$$

assuming that on the “last slice” $\mathcal{L}_O^N S$ satisfy the following estimate,

$$|r^{1-\frac{2}{p}} \mathcal{L}_O^N S|_{p,S}(\lambda, \nu_*) \leq c \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)}}{\rho_{0,1}^N} \right) , \quad (9.83)$$

then the following inequality holds on \mathcal{K} ,

$$|r^{1-\frac{2}{p}} \mathcal{L}_O^N S|_{p,S}(\lambda, \nu) \leq c \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)}}{\rho_{0,1}^N} \right) . \quad (9.84)$$

Corollary 9.9 *Under the assumptions of Theorem 9.8 the following estimates hold on \mathcal{K} ,*

$$\begin{aligned} |r^{1-\frac{2}{p}} (\mathcal{L}_O^N e_A)^C|_{p,S} &= |r^{(1-\frac{2}{p})} \mathbf{g}(e_C, \mathcal{L}_O^N e_A)|_{p,S} \leq c \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^N} \right) . \\ r^{1-\frac{2}{p}} |(\mathcal{L}_O^{N-1} e_A)^C|_{\infty,S} &= |r^{(1-\frac{2}{p})} \mathbf{g}(e_C, \mathcal{L}_O^N e_A)|_{p,S} \leq c \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^N} \right) . \end{aligned} \quad (9.85)$$

The proof is in subsection 14.10

As second step we estimates, in the same way as done in lemma 4.2, inverting the estimates 15.27 the quantities $r^{1-\frac{2}{p}}(\nabla_O^{N-1}e_A)^C$, namely we have, with c a suitable constant.

Corollary 9.10

$$\begin{aligned} |r^{1-\frac{2}{p}}(\nabla_O^N e_A)^C|_{p,S} &= |r^{(1-\frac{2}{p})}\mathbf{g}(e_C, \mathcal{L}_O^N e_A)|_{p,S} \leq c \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^N} \right) \\ |r^{1-\frac{2}{p}}(\nabla_O^{N-1} e_A)^C|_{\infty,S} &= |r^{(1-\frac{2}{p})}\mathbf{g}(e_C, \mathcal{L}_O^N e_A)|_{p,S} \leq c \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^N} \right). \end{aligned} \quad (9.86)$$

As third and last step we recover the estimates for $\mathbf{g}(e_E, \nabla^N e_A)$ from estimates for $\mathbf{g}(e_C, \nabla_O^N e_A)$. To do this we have substantially to invert lemma 4.1, hence we introduce the quantities $h_{(i),A}$ and $h_{(i),B}$, such that

$$\sum_{i=1}^3 h_{(i),A} {}^iO = e_A \quad , \quad \sum_{i=1}^3 h_{(i),B} {}^iO = e_B \quad (9.87)$$

Exploiting the equations for S and the Leibnitz rule we have the equation

$$\nabla_{(\cdot)} h_{(i),A} = \mathbf{g}(\nabla_{(\cdot)}, e_A) - \mathbf{g}(\nabla_{(\cdot)} {}^{(i)}O) = S_{(\cdot),A} - \mathbf{g}(\nabla_{(\cdot)} {}^iO) \quad (9.88)$$

end hence we have the transport equation for $h_{(i)}$

$$\frac{\partial}{\partial \nu} h_{(i)} = \frac{\Omega \text{tr} \chi}{2} S + \Omega \hat{\chi} \cdot S - \Omega \chi {}^{(i)}O \quad (9.89)$$

Hence, exploiting the previous estimates for ${}^{(i)}O$ and S , see 9.84 and 9.90, we can estimate h on $C_0 \cup \underline{C}_0$ and consequently on all the region. We have the

Theorem 9.11 *Assuming the norm estimates for the null Riemann component of Theorem 4.1, assuming the norm estimates 3.31 for the initial data connection coefficients, the norm estimates 3.29 for the connection coefficients with $J < N$ and the following estimate for h , with $J < N$, c a suitable constant,*

$$|r^{(1-\frac{2}{p})}\mathcal{L}_O^J h|_{p,S} \leq c \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right),$$

then the following inequality holds on \mathcal{K} ,

$$|r^{1-\frac{2}{p}}\mathcal{L}_O^N h|_{p,S}(\lambda, \nu) \leq c \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)}}{\rho_{0,1}^N} \right). \quad (9.90)$$

$$|r^{1-\frac{2}{p}}\mathcal{L}_O^{N-1} h|_{\infty,S}(\lambda, \nu) \leq c \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)}}{\rho_{0,1}^N} \right). \quad (9.91)$$

Corollary 9.12 *Under the assumptions of Theorem 9.6 the following estimates hold on \mathcal{K} , with c a suitable constant,*

$$|r^{1-\frac{2}{p}} \nabla_O^N h|_{p,S}(\lambda, \nu) \leq c \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)}}{\rho_{0,1}^N} \right). \quad (9.92)$$

$$|r^{1-\frac{2}{p}} \nabla_O^{N-1} h|_{\infty,S}(\lambda, \nu) \leq c \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)}}{\rho_{0,1}^N} \right). \quad (9.93)$$

The proof is also in this case a repetition of lemma 4.2 and we do not report it here.

Now in order to estimate $|\nabla_B^N e_A|_{\infty,S}$ we notice that $\nabla_B e_A = \nabla_{h_{(i),B}^{(i)} O} e_A$, hence, we can exploit the estimates of corollary 9.12 and lemma 4.1 with ∇_O instead of ∇_B and $h_{(i),B}$ instead of $^{(i)}O$. to prove the following

Theorem 9.13 *Assuming the norm estimates for the null Riemann component of Theorem 4.1, assuming the norm estimates 3.31 for the initial data connection coefficients, the norm estimates 3.29 for the connection coefficients with $J < N$ and the estimate of corollary 9.12 for h , the following estimates hold on \mathcal{K} ,*

$$|r^{1-\frac{2}{p}} \nabla_{(\cdot)}^{N-1} e_A|_{\infty,S}(\lambda, \nu) \leq c \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)}}{\rho_{0,3}^N} \right). \quad (9.94)$$

With $(\cdot) = e_{(A,B)}$ and $\rho_{0,3} < \rho_{0,1}$, c a suitable constant.

In the same way we can estimate the ∇_A derivatives of the null Riemann components, and of the connection coefficients:

Corollary 9.14 *Under the assumptions of theorem 9.13, the following estimates hold on \mathcal{K} ,⁸⁴*

$$\begin{aligned} |r^{\phi(\mathcal{O})} \nabla_{(\cdot)}^{N-1} \mathcal{O}|_{\infty,S}(\lambda, \nu) &\leq c \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)}}{\rho_{0,3}^N} \right) \\ |r^{\phi(R)-\frac{2}{p}} \nabla_{(\cdot)}^{N-1} R|_{p,S}(\lambda, \nu) &\leq c \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)}}{\rho_{0,3}^N} \right) \end{aligned} \quad (9.95)$$

With $(\cdot) = e_{(A,B)}$ and $\rho_{0,3} < \rho_{0,1}$ and c a suitable constant.

⁸⁴Recall that the sup norm of the J derivative is estimated in term of the $|\cdot|_{p,S}$ norm of the J and of the $J+1$ derivative. In fact the second line is

$$|r^{\phi(\mathcal{O})} \nabla^J \mathcal{O}|_{\infty,S} \leq C \left[\frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)}}{\rho_{0,3}^J} + \frac{(J+1)!}{(J+1)^\alpha} \frac{e^{(J-1)\delta_0} e^{(J-1)\Gamma_0(\lambda)}}{\rho_{0,3}^{(J+1)}} \right] \leq C \frac{(J+1)!}{(J+1)^\alpha} \frac{e^{(J-1)\delta_0} e^{(J-1)\Gamma_0(\lambda)}}{\rho_{0,3}^{(J+1)}}$$

and we neglect $c \geq 1 + \frac{\rho_{0,3}}{(J+1)e^{\delta_0 + \Gamma_0(\lambda)}}$.

9.3 Proof of theorem 3.6

In order to prove theorem 3.6 we have to retrace all the steps of section 15.5, in order to do this we have to obtain the results of lemma 10.3 but now for the connection coefficients. This can be obtained applying lemma 4.2 to the estimates 3.27, see corollary 9.5. Hence we can state the

Corollary 9.15 *Assuming the estimates 3.27, and the estimates of corollary 9.5.*

$$|r^{\phi(\mathcal{O})-\frac{2}{p}}\mathcal{L}_O^J\{\mathcal{O},\underline{\mathcal{O}}\}|_{p,S}(\lambda,\nu) \leq c \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)}}{\rho_{0,1}^J} \right). \quad (9.96)$$

for $J < N$, the following estimates hold on \mathcal{K} , with c a suitable constant,

$$|r^{\phi(\mathcal{O})-\frac{2}{p}}\nabla_O^N\{\mathcal{O},\underline{\mathcal{O}}\}|_{p,S}(\lambda,\nu) \leq c \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)}}{\rho_{0,1}^N} \right). \quad (9.97)$$

The proof follows, as in corollary 9.5, from Theorem 4.2.

Once we have stated the estimates 9.97 we can perform the steps of lemma 10.4 and, exploiting inequalities 9.94, obtain the estimates:

$$|r^{N+\phi(\mathcal{O})-\frac{2}{p}}\nabla^N\{\mathcal{O},\underline{\mathcal{O}}\}|_{p,S}(\lambda,\nu) \leq c \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)}}{\rho^N} \right). \quad (9.98)$$

Hence we can consider proved the theorem 3.6.

9.4 Proof of corollary 3.7

In order to estimate $|r^{(\cdot)}\mathcal{L}_O^h\mathcal{O}|_\infty(\lambda,\nu)$ with $h \leq N-1$ and $|r^{(\cdot)+1}\mathcal{L}_O^h\nabla\mathcal{O}|_\infty(\lambda,\nu)$ with $h \leq N-2$ we recall propositions 3.3.3 and 3.3.4 of [Ch-Kl] for a 1-form and symmetric 2-tensor, let us recall them.

Proposition

Let U_α be 1-form, then it does exists a constant C such that

$$|\nabla U|^2 \leq C \sum_i |\mathcal{L}_{(i)}\mathcal{O}U|^2 \quad (9.99)$$

Let $U_{\alpha\beta}$ be a symmetric two tensor, then it does exists a constant C such that

$$|\nabla U|^2 \leq C \sum_i |\mathcal{L}_{(i)}\mathcal{O}U|^2 \quad (9.100)$$

We will proceed by induction. Moreover from the Sobolev lemma

$$|r^{(\cdot)}\mathcal{L}_O^h\mathcal{O}|_\infty(\lambda,\nu) \leq C \left[|r^{(\cdot)-\frac{2}{4}}\mathcal{L}_O^h\mathcal{O}|_{4,S}(\lambda,\nu) + |r^{(\cdot)+1-\frac{2}{4}}\nabla\mathcal{L}_O^h\mathcal{O}|_{4,S}(\lambda,\nu) \right] \quad (9.101)$$

Hence, to estimate $|r^{(\cdot)} \mathcal{L}_O^h \mathcal{O}|_\infty(\lambda, \nu)$, we have to estimate $|r^{(\cdot)-\frac{2}{4}} \mathcal{L}_O^h \mathcal{O}|_{4,S}$ and $|r^{((\cdot)+1-\frac{2}{4})} \nabla \mathcal{L}_O^h \mathcal{O}|_{4,S}$. The first term can be estimated inductively while for the second, it holds

$$\begin{aligned} |\nabla \mathcal{L}_O^h \mathcal{O}|_{4,S} &\leq |\mathcal{L}_O^h \nabla \hat{\chi} - [\mathcal{L}_O^h, \nabla] \mathcal{O}|_{4,S} \\ &\leq |\mathcal{L}_O^h \nabla \mathcal{O}|_{4,S} + |[\mathcal{L}_O^h, \nabla] \mathcal{O}|_{4,S} \end{aligned} \quad (9.102)$$

and both terms can be estimated inductively. Hence we obtain

$$|r^{(\cdot)+1} \mathcal{L}_O^{h-1} \mathcal{O}|_\infty(\lambda, \nu) \leq C_0 e^{(h-2)\delta} e^{(h-2)\Gamma(\lambda)} \frac{h!}{h^\alpha} \frac{1}{\rho_{0,1}^h}. \quad (9.103)$$

Let us now estimate $|r^{(\cdot)} \mathcal{L}_O^h \nabla \mathcal{O}|_\infty$:

$$\begin{aligned} |r^{(\cdot)+1} \mathcal{L}_O^h \nabla \mathcal{O}|_\infty &\leq |r^{(\cdot)+1} \nabla \mathcal{L}_O^h \mathcal{O}|_{\infty,S} + |r^{(\cdot)+1} [\nabla, \mathcal{L}_O^h] \mathcal{O}|_{\infty,S} \\ &\leq C |r^{(\cdot)+1} \mathcal{L}_O^{h+1} \mathcal{O}|_{\infty,S} + |r^{(\cdot)} [\nabla, \mathcal{L}_O^h] \mathcal{O}|_{\infty,S} \end{aligned} \quad (9.104)$$

Where in the second line we used the inequalities 14.139 or 14.142. Now it is sufficient to observe that the first term can be estimated in the same way of $|\mathcal{L}_O^k \mathcal{O}|_{\infty,S}$ while the second term can be estimated inductively as the $[\nabla, \mathcal{L}_O^h]$ commutator involve only \mathcal{L}_O derivatives up to order $h-1$.

Hence, finally, we obtain the estimate

$$|r^{(\cdot)} \mathcal{L}_O^{k-2} \nabla \mathcal{O}|_\infty(\lambda, \nu) \leq C e^{(k-2)\delta} e^{(k-2)\Gamma(\lambda)} \frac{k!}{k^\alpha} \frac{1}{\rho_{0,1}^k}.$$

9.5 The definition of the canonical foliation

We shortly motivated in subsection 3.3 the introduction of the “double null cone canonical foliation” to prove our result; we add here a more detailed discussion to show why it is needed and the way it is built.

From equation 9.40 in Lemma 9.4 it is immediate to see that the estimate for $\tilde{\mu}$ and subsequently the one for η have a worst decay than expected, namely it follows that, integrating from \underline{C}_0 we could only prove,

$$\left| r^{1-\frac{2}{p}} \mathcal{L}_O^{N-1} \tilde{\mu} \right|_{p,S}(\lambda, \nu) \leq \tilde{C}_3 e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N}, \quad (9.105)$$

$$\left| r^{1-\frac{2}{p}} \mathcal{L}_O^N \eta \right|_{p,S}(\lambda, \nu) \leq C_3 e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N}. \quad (9.106)$$

provided that in all the previous proofs no problem is encountered assuming for $J < N$,

$$|r^{1-\frac{2}{p}} \mathcal{L}_O^J \zeta|_{p,S}(\lambda, \nu) \leq C_3 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} . \quad (9.107)$$

This result is not sufficient as it is not difficult to realize in the course of the proof of Theorem 3.6 that, to obtain that the right estimates for the order N angular derivatives of all the connection coefficients are all consistent, we are forced to assume the appropriate (and optimal) decays for all of them together with their derivatives up to order $N - 1$. To obtain this decay we are, therefore, forced to integrate the transport equations, for the not underlined connection coefficients, from above which requires to know in advance their estimates on \underline{C}_* . This can be done provided we choose an appropriate foliation on this last slice proceeding as in the following steps,

Step 1: We assume the previous inductive assumptions for $J < N$ hold in the region $\mathcal{K}(\Lambda_a, \Pi_a)$, see (I), where we have the analytic solution.

Step 2: We assume that there are initial data for Ω on \underline{C}_0 such that on the last slice $\underline{C}_* \equiv \underline{C}(\nu_* = \Pi_a)$ the following equation is satisfied

$$\tilde{\mu}|_{\underline{C}_*} = \frac{1}{4}(\text{tr}\chi \text{tr}\underline{\chi} - \overline{\text{tr}\chi \text{tr}\underline{\chi}}) . \quad (9.108)$$

Observe that, recalling the definition of $\tilde{\mu}$, see 9.37, this equation is equivalent to the following one on \underline{C}_* ,

$$\Delta \log \Omega = -\text{div} \zeta + \frac{1}{2}(\hat{\chi} \hat{\underline{\chi}} - \overline{\hat{\chi} \hat{\underline{\chi}}}) - (\rho - \bar{\rho}) . \quad (9.109)$$

Let us denote $\Omega_* \equiv \Omega|_{\underline{C}_*}$ the solution of 9.109 .

Step 3: Provided we know the estimates for the $|\cdot|_{p,S}$ norms of $\tilde{\mu}|_{\underline{C}_*} = \frac{1}{4}(\text{tr}\chi \text{tr}\underline{\chi} - \overline{\text{tr}\chi \text{tr}\underline{\chi}})$ and for its angular derivatives \mathcal{L}_O^J with $J \leq N - 1$, we can integrate the equation for $\tilde{\mu}$ from the last slice going down which implies that we can gain an extra r decay factor. This allows to conclude that the following estimates hold

$$\begin{aligned} |r^{2-\frac{2}{p}} \mathcal{L}_O^{N-1} \tilde{\mu}|_{p,S}(\lambda, \nu) &\leq \tilde{C}_3 e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} , \\ |r^{2-\frac{2}{p}} \mathcal{L}_O^N \eta|_{p,S}(\lambda, \nu) &\leq C_3 \log N e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} . \end{aligned} \quad (9.110)$$

Step 4: Integrating from above the equation for $\tilde{\omega}$, again knowing the estimates for the $|\cdot|_{p,S}$ norms of $\tilde{\omega}|_{\underline{C}_*}$ and its angular derivatives ∇^J with $J \leq N - 2$, it follows that all the remaining estimates for the angular derivatives of the connection coefficients can be obtained, those for χ integrating from \underline{C}_* and those from the underlined quantities integrating from C_0 . This result requires

that the solution Ω_* does exist. Once this has been proved the existence of the appropriate initial data on \underline{C}_0 follows observing that

$$\Omega(\lambda, \nu_0) = \Omega(\lambda, \nu_*) + 2 \int_{\nu_0}^{\nu_*} d\nu \Omega^2 \omega \quad (9.111)$$

and the integral is finite due to the asymptotic behaviour of ω .

Summarizing, to prove Lemma 3.1 we need to achieve the following preliminary results:

- a) The proof that on the whole \underline{C}_* the solution Ω_* exists.
- b) The proof that we control on the whole \underline{C}_* the $|\cdot|_{p,S}$ norms of $\tilde{\mu}$ and of $\tilde{\omega}$ and their angular derivatives with $J \leq N-1$ and $J \leq N-2$ respectively. Let us discuss how to obtain these results.

Step 5: To prove that the solution Ω_* exists one proceeds, basically, as in [Kl-Ni2] and [Ni]. We prove the existence of the solution Ω_* on \underline{C}_* in two steps: first we prove that a solution, Ω , exists in a small interval of \underline{C}_* , $\underline{C}_*([\lambda_0, \lambda_0 + \delta])$ and that in this interval $\mathcal{L}_O^J \eta$, $J \leq N$ satisfies the following bounds, with a given \tilde{C}_0 ,

$$|r^{(J+2)-\frac{2}{p}} \mathcal{L}_O^J \eta|_{p,S}(\lambda, \nu) \leq \tilde{C}_0 \log J e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} ; \quad (9.112)$$

we consider then the solution Ω in the largest possible interval of \underline{C}_* , $\underline{C}_*([\lambda_0, \lambda_1])$ where 9.112 is still valid. We prove that, in fact, in this interval the previous bounds hold with \tilde{C}_0 substituted by $C_3 \leq \frac{\tilde{C}_0}{2}$. This implies that the estimate 9.112 and the solution holds on the whole \underline{C}_* . For it we have to prove, see next step, that we control $\hat{\chi}$ and $\hat{\underline{\chi}}$ so that the estimates 9.112 can be improved and moreover we have to prove that the estimate for $\rho - \bar{\rho}$ gives a small contribution which is possible to do as in this region⁸⁵ the canonical foliation does exist and, therefore, we can express the Riemann components in terms of the “small” initial data Riemann components.

The local existence has been proved in [Ni], it is easy to extend it to a solution for all its ∇^J derivatives. To find the “global solution again one has to proceed as in [Kl-Ni2] extending the result for all the derivatives $\nabla^J \eta$. Observe that equation 9.109 can be rewritten as

$$\mathring{d}\nu \eta = \frac{1}{2}(\hat{\chi}\hat{\underline{\chi}} - \overline{\hat{\chi}\hat{\underline{\chi}}}) - (\rho - \bar{\rho}) = \frac{1}{4}(\text{tr}\chi \text{tr}\underline{\chi} - \overline{\text{tr}\chi \text{tr}\underline{\chi}}) + 2(K - \bar{K}) , \quad (9.113)$$

therefore we have to control $\chi, \underline{\chi}$ and their derivatives up to $N-1$ on $\underline{C}_*([\lambda_0, \lambda_1])$. This can be done looking at the transport equations for these quantities along the incoming cones and observing that, as in this interval Ω_* exists, the transport equation for $\text{tr}\chi$ which usually have a “loss of derivatives” does not have it due to the choice $\Omega = \Omega_*$. This equation is in fact,

$$\mathcal{D}_3 \text{tr}\chi + \text{tr}\underline{\chi} \text{tr}\chi - 2\omega \text{tr}\chi - 2\mathring{d}\nu \eta - 2|\eta|^2 + 2K = 0 \quad (9.114)$$

⁸⁵Bounded by $C(\lambda_0)$, $C(\lambda_0 + \delta)$, $C(\nu_0; [\lambda_0, \lambda_0 + \delta])$ and $C(\nu_*; [\lambda_0, \lambda_0 + \delta])$.

and it becomes on \underline{C}_* , due to 9.113,

$$\mathfrak{D}_3 \Omega \operatorname{tr} \chi + \frac{1}{2} \Omega \operatorname{tr} \underline{\chi} \operatorname{tr} \chi - 2\Omega |\eta|^2 + \Omega \overline{\hat{\chi} \hat{\chi}} - 2\Omega \bar{\rho} = 0 ; \quad (9.115)$$

it is immediate to recognize that there is no loss of derivatives for the angular derivatives of $\operatorname{tr} \chi$, for $\nabla^J \operatorname{tr} \chi$, as $\nabla \bar{\rho} = 0$. The situation is different for $\hat{\chi}$ whose transport equation

$$\mathfrak{D}_3 \hat{\chi} + \frac{1}{2} \operatorname{tr} \underline{\chi} \hat{\chi} + \frac{1}{2} \operatorname{tr} \chi \underline{\hat{\chi}} - 2\underline{\omega} \hat{\chi} - \nabla \hat{\otimes} \eta - \eta \hat{\otimes} \eta = 0 ,$$

transforms into

$$\mathfrak{D}_3 \Omega \hat{\chi} + \frac{1}{2} \Omega \operatorname{tr} \underline{\chi} \hat{\chi} + \frac{1}{2} \Omega \operatorname{tr} \chi \underline{\hat{\chi}} - \Omega \nabla \hat{\otimes} \eta - \Omega \eta \hat{\otimes} \eta = 0 .$$

This equation has still a loss of derivatives, but in fact is not required to control the last slice.⁸⁶ In fact to control η we need to control $\tilde{\mu}|_{\underline{C}_*}$, 9.108, which requires to control on \underline{C}_* only $\operatorname{tr} \chi$ (and of course $\operatorname{tr} \underline{\chi}$).

To control the angular derivatives of $\underline{\omega}$ we have to control on \underline{C}_* , $\nabla^{N-2} \mathfrak{d}iv \hat{V}$ where $\hat{V} = (\tilde{V} - \Omega \underline{\beta})$ and $\tilde{V} = 2\nabla \Omega \underline{\omega}$; to do it we proceed as follows: we start from equation 9.109 which we rewrite as

$$\mathfrak{A} \log \Omega = \frac{1}{2} \left[\mathfrak{d}iv \underline{\eta} + \frac{1}{2} (\hat{\chi} \hat{\chi} - \overline{\hat{\chi} \hat{\chi}}) - (\rho - \bar{\rho}) \right] . \quad (9.116)$$

Proceeding as in [Kl-Ni2] we differentiate this equation with respect to \mathbf{D}_3 and obtain an elliptic equation, see [Kl-Ni2] eq. (7.4.21),

$$\mathfrak{A}(\Omega \underline{\omega}) = \mathfrak{d}iv F_1 + G_1 - \overline{G}_1 \quad (9.117)$$

and applying to it $\mathcal{L}_O^{N-2} \mathfrak{d}iv$ we obtain the elliptic equation for $\mathcal{L}_O^{N-2} \tilde{\omega}$ whose solution gives the required $\mathcal{L}_O^{N-2} \tilde{\omega}$ on \underline{C}_* which is what we need to complete the proof of Theorem 9.3.

Remarks:

Observe that to solve equation 9.109 on the last slice and the analogous elliptic equations for the tangential derivatives up to N , we need the estimates for the derivatives of $\hat{\chi}$ only up to order $N-1$ which are already provided by the inductive assumptions. Moreover in any case the estimates of $\mathcal{L}_O^N \hat{\chi}$ on \underline{C}_ are provided by the Hodge equations obtained deriving*

$$\mathfrak{d}iv \hat{\chi} = \frac{1}{2} \nabla \operatorname{tr} \chi - \beta - \zeta \cdot \hat{\chi} + \frac{1}{2} \zeta \operatorname{tr} \chi = \frac{\Omega}{2} \psi - \zeta \hat{\chi} - \beta$$

and therefore do not use the transport equation along \underline{C}_ , but, viceversa, require the previous knowledge of the angular derivatives of $\operatorname{tr} \chi$.*

⁸⁶In the sense that we need only lower derivatives, assumed by induction, see the following remark.

What we are left to do is to show that $\nabla^{N-2}\tilde{\omega}|_{\underline{C}_*}$ and $\nabla^{N-1}\tilde{\mu}|_{\underline{C}_*}$ do exist and satisfy the expected estimates. The estimates we need for it are the estimates along \underline{C}_* of $\nabla^N \text{tr}\chi$ which are provided by the transport equation for $\text{tr}\chi$ along \underline{C}_* in the appropriate canonical foliation.

A last remark has to be done about the local solution of equation 9.109 in $\underline{C}_*([\lambda_0, \lambda_0 + \delta])$. There we need a background foliation at our disposal to solve it. Recall that this equation is not linear and that the fixed point solution requires some smallness of the various quantities the connection coefficients and the Riemann components. In particular we need to control the Riemann component ρ in the canonical foliation in terms of those relative to the background foliation. This has been discussed at length in [Kl-Ni2] and [Ni].

9.6 The estimates of the “initial data” on the last slice

To complete the proof of Theorem 9.2 and Theorem 9.3 and, therefore, of Theorem 3.6 we need to control, as required in steps 3 and 4, the estimates on \underline{C}_* of $\mathcal{L}_O^{N-1}\tilde{\mu}$ and $\mathcal{L}_O^{N-2}\tilde{\omega}$, more specifically we have to prove the following,

$$|r^{2-\frac{2}{p}}\mathcal{L}_O^{N-1}\tilde{\mu}|_{p,S}(\lambda, \nu_*) \leq C_6^{(*)} \left(e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \frac{N!}{N^\alpha \rho_{0,1}^N} \right) \quad (9.118)$$

$$|r^{1-\frac{2}{p}}\tau_-\mathcal{L}_O^{N-2}\tilde{\omega}|_{p,S}(\lambda, \nu_*) \leq \tilde{C}_2^{(*)} \left(e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \frac{N!}{N^\alpha \rho_{0,1}^N} \right). \quad (9.119)$$

Sketch of the proof: The steps required to prove these estimates and all the remaining ones on the last slice are the following: we do not give all the details as they are a repetition of the previous discussion:

0) We assume inductively all the connection coefficients angular derivatives up to $J \leq N-1$.

1) To control $\mathcal{L}_O^{N-1}\tilde{\mu}$ on \underline{C}_* we have to control $\mathcal{L}_O^{N-1}\text{tr}\chi$ and $\mathcal{L}_O^{N-1}\text{tr}\underline{\chi}$, already at disposal by inductive assumptions.

2) From the definition of $\tilde{\mu}|_{\underline{C}_*}$ it follows that to control $\mathcal{L}_O^N\eta$ we need $\mathcal{L}_O^{N-1}\hat{\chi}$, $\mathcal{L}_O^{N-1}\hat{\underline{\chi}}$, $\mathcal{L}_O^{N-1}(\rho, \sigma)$, the first two by inductive assumptions, the last two via the control of the \mathcal{Q} norms.

3) From the equation Ω_* satisfies on \underline{C}_* it follows that $\mathcal{L}_O^N \log \Omega_*$ is under control once we control $\mathcal{L}_O^{N-1}\zeta$, $\mathcal{L}_O^{N-1}\hat{\chi}$, $\mathcal{L}_O^{N-1}\hat{\underline{\chi}}$, $\mathcal{L}_O^{N-1}\rho$, therefore once we control $\mathcal{L}_O^N \log \Omega_*$ we control $\mathcal{L}_O^N\zeta$ and $\mathcal{L}_O^N\eta$.

4) To control $\underline{\omega}$ on \underline{C}_* we apply \mathcal{D}_3 to the elliptic equation satisfied by $\log \Omega$ and we control $\mathcal{L}_O^N \underline{\omega}$ as we already control $\mathcal{L}_O^N \underline{\eta}$ and the remaining quantities are known by inductive assumptions or through the \mathcal{Q} norms.

5) We are left to control $\mathcal{L}_O^N \text{tr}\hat{\chi}$ and $\mathcal{L}_O^N \hat{\chi}$, the second one is controlled by the Hodge equations for $\mathcal{L}_O^{N-1}\hat{\chi}$ which at their turn require the control of

$\mathcal{L}_O^N \text{tr} \chi$; therefore this is the norm estimate we are left to control and this is done through the transport equation 9.115 where, as we are on \underline{C}_* , there is no loss of derivatives.

9.6.1 The initial data and the “last slice problem”

A contradiction seems to appear as we claim that to prove the analytic solution existence, we assign initial data on $C_0 \cup \underline{C}_0$ and, nevertheless, to prove the norm estimates of Lemma 3.1 we have to assign “final data” on the last slice $\underline{C}(\nu_*)$. Nevertheless this contradiction is only apparent and the picture of the global strategy to prove the existence of a global analytic solution goes in the following way:

We assign initial data on $C_0 \cup \underline{C}_0$ satisfying the appropriate norm estimates 3.31 and 3.32 and applying Cauchy-Kowalevski we prove, as discussed in (I), the existence of the solution in a small region; then we denote, see Theorem 3.1, $\mathcal{K}(\Lambda_a, \Pi_a) \equiv \{(\lambda, \nu) \in [\nu_0, \Lambda_a] \times [\lambda_0, \Pi_a]\}$ as the larger region where, with the assigned initial data, this analytic solution does exist. Observe that up to now the double null foliation of the existence region is not specified.

Introducing on the upper part $\underline{C}(\Pi_a)$ of the boundary of $\mathcal{K}(\Lambda_a, \Pi_a)$, the part we call in the previous subsection “the last slice” and denoted $\underline{C}(\nu_*)$, an appropriate foliation defined through Ω_* , we prove, starting from $S(\lambda_0, \nu_*) \subset C_0$ and exploiting the energy norms from \underline{C}_0 , appropriate norm estimates for the not underlined connection coefficients and using them and the transport equations on the outgoing cones we prove the norm estimates of Theorem 3.1; this is the crucial step to show that the region $\mathcal{K}(\Lambda_a, \Pi_a)$ can be extended implying, to avoid a contradiction, that this region is in fact unbounded.

Of course this also implies that the double null foliation of this region required to prove its unboundedness is, see also [Kl-Ni2], the “double null canonical foliation”, that is the one determined by the foliation imposed on the last slice to obtain the correct “final data norms”.

Finally the last thing to observe is that, going back from the data on the last slice with the outgoing transport equations to \underline{C}_0 we obtain some norms bounds that the not underlined connection coefficients have to satisfy on \underline{C}_0 and we have to prove that these bounds are compatible with the previously assumed initial data; this is easy to prove for the following reasons: we can now assume all the transport equations along the outgoing cones without worrying anymore of loss of derivatives as we already have the estimates for all order derivatives and due to the fact that the final parameter ρ appearing in these estimates is smaller than the initial one, ρ_0 , it is immediate to realize that all these estimates are consistent with the initial data estimates.⁸⁷

⁸⁷In principle the foliation on \underline{C}_0 of the initial data is different from the foliation induced on \underline{C}_0 from the canonical foliation, but it can be proved, proceeding as in [Kl-Ni2] that, due to the fact that we are considering a “small initial data problem” the two induced foliations are near and the result follows.

10 The complete results of Section 4

10.1 The detailed steps in the proof of Theorem 4.1.

The proof of this theorem has been discussed in subsection 4.2 where the various steps of the proof are listed, here we give more details for each of them.

10.1.1 Step 1: Initial data estimates for $\nabla_O^{J-1}\Psi$

We start with the initial data, see 3.15 and 3.16, from them and the expression of the Riemann components in terms of the connection coefficients, see for example [Kl-Ni2] chapter 3, we obtain immediately the following estimates, where with $\Psi(R)$ we denote a generic null Riemann component and with $\hat{C}_{0,0}$ a suitable common constant,

$$||u|_{\underline{C}}^{\phi(\Psi)} r^{\phi(\Psi)+\epsilon+(J-1)-\frac{2}{p}} \nabla_O^{J-1}\Psi(R)|_{p,S} \leq \hat{C}_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)}}{\rho_0^J}, \quad (10.1)$$

where ρ_0 is the “convergence radius” for the initial data and $\phi(\Psi)$ has been defined in 4.9.

Remark:

We use here and in the following the convention that if the constant of the connection coefficient or the Riemann null component estimates are multiplied by a constant c which does not depend on these estimates nor on the combinatorial sums we omit it.

As sketched in subsection 4.2 we move from $\nabla_O^{J-1}\Psi(R)$ to $\nabla_O^{J-1}\Psi(R)$, proving the following lemma:

Lemma 4.1 *Under the previous estimates for the connection coefficients and for the null Riemann components on $C_0 \cup \underline{C}_0$, the following estimates for the ∇_O derivatives of the null Riemann components hold:*

$$||\lambda|_{\underline{C}}^{\phi(\Psi)} r^{\phi(\Psi)+\epsilon+(J-1)-\frac{2}{p}} \nabla_O^{J-1}\Psi(R)|_{p,S} \Big|_{C_0 \cup \underline{C}_0} \leq |O|_\infty^{J-1} \hat{C}_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)}}{\rho_{0,1}^J}, \quad (10.2)$$

where $\rho_{0,1} < \rho_0$ and $|O|_\infty$ is the sup norm over the whole initial cones .⁸⁸

Proof: See appendix to section 10.

Proceeding exactly in the same way we also prove the following corollary, Moving from the estimate for $\nabla_O^{J-1}\Psi(R)$ to the estimate for $\Psi(\hat{\mathcal{L}}_O^{J-1}R)$ requires some intermediate steps: first we go from $\nabla_O^{J-1}\Psi(R)$ to $\mathcal{L}_O^{J-1}\Psi(R)$, then from $\mathcal{L}_O^{J-1}\Psi(R)$ to $\Psi(\mathcal{L}_O^{J-1}R)$ and finally from $\Psi(\mathcal{L}_O^{J-1}R)$ to $\Psi(\hat{\mathcal{L}}_O^{J-1}R)$. The following lemma is proved in the appendix to section 10.

⁸⁸Remember that, as $O = \sum_{i=1}^3 O^{(i)}$ we have to consider also a factor 3^{J-1} in the $|O|_\infty^{J-1}$ norms. We do not write explicitly it to avoid cumbersome notations.

Lemma 4.2 *Under the previous assumptions on the (initial data) connection coefficients and the previous estimates for $\nabla_O^{J-1}\Psi(R)$, the following bounds hold,*

$$\left\| |\lambda|^{\phi(\Psi)} r^{\phi(\Psi)+\epsilon-\frac{2}{p}} \mathcal{L}_O^{J-1}\Psi(R)|_{p,S} \right\|_{C_0 \cup \underline{C}_0} \leq \hat{C}_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)}}{\rho_{0,1}^J} \quad (10.3)$$

moreover

$$\left\| |\lambda|^{\phi(\Psi)} r^{\phi(\Psi)+\epsilon-\frac{2}{p}} \Psi(\mathcal{L}_O^{J-1}R)|_{p,S} \right\|_{C_0 \cup \underline{C}_0} \leq \hat{C}_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)}}{\rho_{0,1}^J} \quad (10.4)$$

and finally

$$\left\| |\lambda|^{\phi(\Psi)} r^{\phi(\Psi)+\epsilon-\frac{2}{p}} \Psi(\hat{\mathcal{L}}_O^{J-1}R)|_{p,S} \right\|_{C_0 \cup \underline{C}_0} \leq \hat{C}_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)(\nu)}}{\rho_{0,1}^J}, \quad (10.5)$$

where $\rho_{0,1} < \rho_0$

10.1.2 Step 2: The initial data energy type norms

Lemma 10.1 *From the previous estimates for the null Riemann components, we derive the following estimates for the $\mathcal{Q}_{(0),2}^{(J-2)}(\lambda_0, \nu)$ and $\underline{\mathcal{Q}}_{(0),2}^{(J-2)}(\lambda, \nu_0)$, part of the $\mathcal{Q}_0^{(J-2)}$ norms,⁸⁹⁹⁰*

$$\mathcal{Q}_{(0),2}^{(J-2)}(\lambda_0, \nu) + \underline{\mathcal{Q}}_{(0),2}^{(J-2)}(\lambda, \nu_0) \leq \left(C^{(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma(\lambda)}}{\rho_{0,1}^J} \right)^2. \quad (10.6)$$

Proof: Recalling, see definition 15.5, the explicit expressions,

$$\begin{aligned} \mathcal{Q}_{(0),2}^{(J-2)}(\lambda_0, \nu) &\equiv \int_{C(\lambda_0; [\nu_0, \nu])} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T (\hat{\mathcal{L}}_O^{J-2} W))(\bar{K}, \bar{K}, \bar{K}, e_4) \\ &\quad + \int_{C(\lambda_0; [\nu_0, \nu])} Q(\hat{\mathcal{L}}_O^2 (\hat{\mathcal{L}}_O^{J-2} W))(\bar{K}, \bar{K}, T, e_4) \\ &\quad + \int_{C(\lambda_0; [\nu_0, \nu])} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T (\hat{\mathcal{L}}_O^{J-2} W))(\bar{K}, \bar{K}, \bar{K}, e_4) \\ \underline{\mathcal{Q}}_{(0),2}^{(J-2)}(\lambda, \nu_0) &\equiv \int_{\underline{C}(\nu_0; [\lambda_0, \lambda])} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T (\hat{\mathcal{L}}_O^{J-2} W))(\bar{K}, \bar{K}, \bar{K}, e_3) \\ &\quad + \int_{\underline{C}(\nu_0; [\lambda_0, \lambda])} Q(\hat{\mathcal{L}}_O^2 (\hat{\mathcal{L}}_O^{J-2} W))(\bar{K}, \bar{K}, T, e_3) \\ &\quad + \int_{\underline{C}(\nu_0; [\lambda_0, \lambda])} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T (\hat{\mathcal{L}}_O^{J-2} W))(\bar{K}, \bar{K}, \bar{K}, e_3) \end{aligned}$$

⁸⁹We omit the $\mathcal{Q}_{(0),1}^{(J-2)}$ norms

⁹⁰($\varepsilon_{C(0,4)}$) instead of $C_{(0,4)}$ if $J < J_0$.

we look at the second term of $\mathcal{Q}_{(0),2}^{(J-2)}(\lambda_0, \nu)$, all the remaining terms are treated, basically, in the same way. Recalling the explicit expression of $Q(W)(\bar{K}, \bar{K}, T, e_4)$, see [Kl-Ni2] Chapter 3, eq. (3.5.1),

$$Q(W)(\bar{K}, \bar{K}, T, e_4) = \frac{1}{4}\nu^4|\alpha(W)|^2 + \frac{1}{2}(\nu^4 + 2\nu^2\lambda^2)|\beta(W)|^2 + \frac{1}{2}(\lambda^4 + 2\nu^2\lambda^2)(\rho(W)^2 + \sigma(W)^2) + \frac{1}{2}\lambda^4|\underline{\beta}(W)|^2$$

we have the following inequality,

$$\begin{aligned} \int_{C(\lambda_0; [\nu_0, \nu])} Q(\hat{\mathcal{L}}_O^H W)(\bar{K}, \bar{K}, T, e_4) &= \frac{1}{2} \int_{\nu_0}^{\nu} d\nu' \int_{S(\lambda_0, \nu')} \left[\frac{1}{2}\nu'^4 |\alpha(\hat{\mathcal{L}}_O^H W)|^2 + (\nu'^4 + 2\nu'^2\lambda_0^2) |\beta(\hat{\mathcal{L}}_O^H W)|^2 + \right. \\ &\quad \left. + (\lambda_0^4 + 2\nu'^2\lambda_0^2)(\rho(\hat{\mathcal{L}}_O^H W)^2 + \sigma(\hat{\mathcal{L}}_O^H W)^2) + \lambda_0^4 |\underline{\beta}(\hat{\mathcal{L}}_O^H W)|^2 \right] \\ &\leq c \int_{\nu_0}^{\nu} d\nu' \left[|\nu'^2 \alpha(\hat{\mathcal{L}}_O^H W)|_{(2, S(\lambda_0, \nu'))}^2 + |\nu'^2 \beta(\hat{\mathcal{L}}_O^H W)|_{(2, S(\lambda_0, \nu'))}^2 + |\nu' \lambda_0 \rho(\hat{\mathcal{L}}_O^H W)|_{(2, S(\lambda_0, \nu'))}^2 \right. \\ &\quad \left. + |\nu' \lambda_0 \sigma(\hat{\mathcal{L}}_O^H W)|_{(2, S(\lambda_0, \nu'))}^2 + |\lambda_0^2 \underline{\beta}(\hat{\mathcal{L}}_O^H W)|_{(2, S(\lambda_0, \nu'))}^2 \right] \\ &\leq c \int_{\nu_0}^{\nu} d\nu' \left[\frac{1}{\nu'^{1+\epsilon}} |r^{\frac{7}{2}+\epsilon-\frac{3}{2}} \alpha(\hat{\mathcal{L}}_O^H W)|_{(2, S(\lambda_0, \nu'))}^2 + \frac{1}{\nu'^{1+\epsilon}} |r^{\frac{7}{2}+\epsilon-\frac{3}{2}} \beta(\hat{\mathcal{L}}_O^H W)|_{(2, S(\lambda_0, \nu'))}^2 \right. \\ &\quad + \frac{|\lambda_0|}{\nu'^{2+\epsilon}} |\nu'^{3+\epsilon-\frac{3}{2}} \lambda_0|^{\frac{1}{2}} \rho(\hat{\mathcal{L}}_O^H W)|_{(2, S(\lambda_0, \nu'))}^2 + \frac{|\lambda_0|}{\nu'^{2+\epsilon}} |\nu'^{3+\epsilon-\frac{3}{2}} \lambda_0|^{\frac{1}{2}} \sigma(\hat{\mathcal{L}}_O^H W)|_{(2, S(\lambda_0, \nu'))}^2 \\ &\quad \left. + \frac{|\lambda_0|}{\nu'^{1+\epsilon}} |\nu'^{2+\epsilon-\frac{3}{2}} \lambda_0|^{\frac{3}{2}} \underline{\beta}(\hat{\mathcal{L}}_O^H W)|_{(2, S(\lambda_0, \nu'))}^2 \right]. \end{aligned} \quad (10.7)$$

Remarks:

i) The weight for the ρ null component is correct provided $H > 0$, this is the case from the explicit expression of the \mathcal{Q} norms. The non derived ρ has a decay $O(r^{-3})$ which cannot be improved.

ii) As it appears from the previous estimates we could assume a weaker decay for ρ and σ on the initial data (without the ϵ factor) as also in this case the \mathcal{Q} norms are bounded on $C_0 \cup \underline{C}_0$. See the discussion in section 7.

Inserting in 10.7 the estimates 10.5 proved in Lemma 4.2 we obtain

$$\begin{aligned} \int_{C(\lambda_0; [\nu_0, \nu])} Q(\hat{\mathcal{L}}_O^H W)(\bar{K}, \bar{K}, T, e_4) &\leq c \left(\hat{C}_{0,0} \frac{(H+1)!}{(H+1)^\alpha} \frac{e^{(H-1)\delta_0}}{\rho_{0,1}^{H+1}} \right)^2 \int_{\nu_0}^{\nu} \frac{d\nu'}{\nu'^{1+\epsilon}} e^{2(H-1)\Gamma_0(\lambda)} \\ &\leq \left(\hat{C}_{0,0} \frac{(H+1)!}{(H+1)^\alpha} \frac{e^{(H-1)\delta_0} e^{(H-1)\Gamma_0(\lambda)}}{\rho_{0,1}^{H+1}} \right)^2 \int_{\nu_0}^{\nu} \frac{d\nu'}{\nu'^{1+\epsilon}} \\ &\leq \left(\hat{C}_{0,0} \frac{(H+1)!}{(H+1)^\alpha} \frac{e^{(H-1)\delta_0} e^{(H-1)\Gamma_0(\lambda)(\nu)}}{\rho_{0,1}^{H+1}} \right)^2 \\ &\leq \left(\hat{C}_{0,0} \frac{(H+1)!}{(H+1)^\alpha} \frac{e^{(H-1)\delta_0} e^{(H-1)\Gamma_0(\lambda)}}{\rho_{0,1}^{H+1}} \right)^2 \end{aligned} \quad (10.8)$$

Finally we can rewrite the previous estimate as

$$\begin{aligned}
& \int_{C(\lambda_0; [\nu_0, \nu])} Q(\hat{\mathcal{L}}_O^2(\hat{\mathcal{L}}_O^{J-2}W))(\bar{K}, \bar{K}, T, e_4) = \int_{C(\lambda_0; [\nu_0, \nu])} Q(\hat{\mathcal{L}}_O^J W)(\bar{K}, \bar{K}, T, e_4) \\
& \leq \left(\hat{C}_{0,0} \frac{(J+1)!}{(J+1)^\alpha} \frac{e^{(J-1)\delta_0} e^{(J-1)\Gamma_0(\lambda)}}{\rho_{0,1}^{J+1}} \right)^2 \\
& \leq \left(\hat{C}_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta} e^{(J-2)\Gamma_0(\lambda)}}{\rho_{0,1}^J} \right)^2,
\end{aligned} \tag{10.9}$$

with suitable $C^{(2)}$ and $\delta > \delta_0$.

Let us calculate also $\int_{\underline{C}(\nu; [\lambda_0, \lambda])} Q(\hat{\mathcal{L}}_O(\hat{\mathcal{L}}_O^{J-1}W))(\bar{K}, \bar{K}, T, e_3)$, exploiting the estimates of [Kl-Ni2] Chapter 3, we obtain the, as before, estimates

$$\begin{aligned}
& \int_{\underline{C}(\nu; [\lambda_0, \lambda])} Q(\hat{\mathcal{L}}_O^2(\hat{\mathcal{L}}_O^{J-2}W))(\bar{K}, \bar{K}, T, e_3) \leq \left(\hat{C}_{0,0} \frac{(J+1)!}{(J+1)^\alpha} \frac{e^{(J-1)\delta_0}}{\rho_{0,1}^{J+1}} \right)^2 \int_{\lambda_0}^{\lambda} \frac{d\lambda'}{\lambda'^{1+\epsilon}} e^{2(J-1)\Gamma_0(\lambda')} \\
& \leq \left(\hat{C}_{0,0} \frac{(J+1)!}{(J+1)^\alpha} \frac{e^{(J-1)\delta_0} e^{(J-1)\Gamma_0(\lambda)}}{\rho_{0,1}^{J+1}} \right)^2 \int_{\lambda_0}^{\lambda} \frac{d\lambda'}{\lambda'^{1+\epsilon}} e^{2(J-1)(\Gamma_0(\lambda') - \Gamma_0(\lambda))} \\
& \leq' \left(\hat{C}_{0,0} \frac{(J+1)!}{(J+1)^\alpha} \frac{e^{(J-1)\delta_0} e^{(J-1)\Gamma_0(\lambda)}}{\rho_{0,1}^{J+1}} \right)^2 \int_{\lambda_0}^{\lambda} \frac{d\lambda'}{\lambda'^{1+\epsilon}} e^{2(J-1)(\frac{\lambda' - \lambda}{\lambda' \lambda})} \\
& \leq \left(\hat{C}_{0,0} \frac{(J+1)!}{(J+1)^\alpha} \frac{e^{(J-1)\delta_0} e^{(J-1)\Gamma_0(\lambda)}}{\rho_{0,1}^{J+1}} \right)^2 \\
& \leq \left(\hat{C}_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta} e^{(J-2)\Gamma_0(\lambda)}}{\rho_{0,1}^J} \right)^2
\end{aligned} \tag{10.10}$$

Remark:

It is important to recall, see appendix later, that to control $|\hat{\mathcal{L}}_O^{J-1}\Psi(W)|_{p,S}$ we need to control $\mathcal{Q}^{(J-2)}$ and not $\mathcal{Q}^{(J-1)}$.

10.1.3 Step 3: The conservation of the \mathcal{Q} norms

“Step 3” is a generalization of the boundedness of the \mathcal{Q} norms proved in [Kl-Ni2], Chapter 6, see also [Ch-Kl]. The main difference is that we have now a larger family of \mathcal{Q} norms to control all the derivatives (tangential and null) of the Riemann components; this makes the control of these norms technically more complicated, but the final result is that, provided the $\mathcal{Q}^{(0)}$ norms are bounded in a region \mathcal{K} , which can also be unbounded, in terms of the initial data $\mathcal{Q}^{(0)}$ norms, then we control all the \mathcal{Q} norms in the same region.

As we said at the beginning of subsection 4.1 we consider for the moment only the norms needed to control all the angular derivatives, see 15.3 and 15.4 and prove the following crucial result:

Theorem 10.1 *Let us define*

$$\mathcal{H}^{(J-2)}(\lambda, \nu) \equiv \sum_{J=2}^N \left(\mathcal{Q}^{(J-2)}(C(\lambda; [\nu_0, \nu])) + \underline{\mathcal{Q}}^{(J-2)}(\underline{C}(\nu; [\lambda_0, \lambda])) \right) , \quad (10.11)$$

let assume that $\mathcal{H}^{(0)}(\lambda, \nu)$ satisfies the following estimates:

$$\mathcal{H}^{(0)}(\lambda, \nu) < C_0^{(1)} \left(\mathcal{Q}_{(0,2)}^{(0)}(\lambda_0, \nu) + \underline{\mathcal{Q}}_{(0,2)}^{(0)}(\lambda, \nu_0) \right) , \quad (10.12)$$

in \mathcal{K} , with $C_0^{(1)}$ a suitable constant, let assume the initial data satisfy the inequalities 3.31 and 3.32 for all $J \leq N-1$, let assume the inductive assumptions 3.29 for the connection coefficients hold with $J \leq N-1$, then the following estimate holds,

$$\mathcal{H}^{(N-2)}(\lambda, \nu)^{\frac{1}{2}} \leq C^{*(1)} \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^N} \right) , \quad (10.13)$$

with $C_0^{(1)} < C^{(1)}$ a suitable constant.*

Proof: The proof goes by induction, see Appendix to Section 10.

10.1.4 Step 4: The internal estimates of $\Psi(\hat{\mathcal{L}}_O^{J-1}W)$, $\Psi(\mathcal{L}_O^{J-1}W)$ and $\mathcal{L}_O^{J-1}\Psi(W)$

Due to the estimates we have proved for the various \mathcal{Q} norms, see Theorem 10.1, we have immediatly

$$\sqrt{\mathcal{Q}^{(J-2)}(\lambda, \nu) + \underline{\mathcal{Q}}^{(J-2)}(\lambda, \nu)} \leq C^{*(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} . \quad (10.14)$$

The following lemma holds, with $C^{*(1)} < C^{(1)}$,⁹¹⁹²

Lemma 10.2 *Under the estimates 10.14 and the inductive assumptions, up to $J-1$ for the connection coefficients and up to $J-2$ for the null Riemann components, the following estimates hold:*

$$||\lambda|^{\phi(\Psi)} r^{\phi(\Psi) - \frac{2}{p}} \Psi(\hat{\mathcal{L}}_O^{J-1}W)|_{p,S}(\lambda, \nu) \leq C^{(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} . \quad (10.15)$$

Proof: The proof of this lemma is analogous to the one in [Kl-Ni2]. It is discussed in detail in Section 15. we recall its main lines as it is a crucial step in proving our global result. We consider for simplicity just one of the null Riemann component, namely $\beta(\hat{\mathcal{L}}_O^{J-1}W)$.

⁹¹The standard estimate is with $p = 4$, for $p \in [2, 4)$ we need an interpolation.

⁹²Remember we need $C^{(1)}$ to be smaller than the constants estimating the connection coefficients.

We define $\tilde{W} = \hat{\mathcal{L}}_O^{J-1} W$ and consider the \mathcal{Q} norms introduced in [Kl-Ni2], but now relative to the Weyl tensor \tilde{W} instead that W . Namely with these $Q(\tilde{W})$ norms, proceeding as in Chapter 5 of [Kl-Ni2] we control $\|r^3 \nabla \beta(\tilde{W})\|_{L^2(C)}$ and $\|r^3 \mathfrak{D}_4 \beta(\tilde{W})\|_{L^2(C)}$, more precisely, for $J > 7$, recalling Lemma 4.1.2 of [Kl-Ni2], we have,⁹³

$$\begin{aligned} & |r^{\frac{7}{2}-\frac{2}{4}} \beta(\tilde{W})|_{p=4,s}(\lambda, \nu) \\ \leq & |r^{\frac{7}{2}-\frac{2}{4}} \beta(\tilde{W})|_{p=4,s}(\lambda, \nu_0) + \left[\|r^3 \nabla \beta(\tilde{W})\|_{L^2(C(\lambda; [\nu_0, \nu]))} + \|r^3 \mathfrak{D}_{e_4} \beta(\tilde{W})\|_{L^2(C(\lambda; [\nu_0, \nu]))} \right] \\ \leq & |r^{\frac{7}{2}-\frac{2}{4}} \beta(\tilde{W})|_{p=4,s}(\lambda, \nu_0) + \sqrt{\mathcal{Q}^{(J-2)}(\lambda, \nu) + \underline{\mathcal{Q}}^{(J-2)}(\lambda, \nu)} \quad (10.16) \\ \leq & \hat{C}_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0 + \underline{\Gamma}_0(\lambda))}}{\rho_0^J} + C^{(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta + \underline{\Gamma}(\lambda))}}{\rho_{0,1}^J} \leq C^{(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta + \underline{\Gamma}(\lambda))}}{\rho_{0,1}^J}, \end{aligned}$$

where the term $|r^{\frac{7}{2}-\frac{2}{4}} \beta(\tilde{W})|_{p=4,s}(\lambda, \nu_0)$ has been estimated using the initial data assumptions for the connection coefficients and its derivatives.

10.1.5 Step 5: Internal estimate for $\nabla^{J-1} \Psi(W)$

The last step is moving from $\Psi(\hat{\mathcal{L}}_O^{J-1} W)$ to $\nabla^{J-1} \Psi(W)$. This is the content of the following lemmas

Lemma 10.3 *Assuming the estimates for the null Riemann components of lemma 10.2,*

$$|\Psi(\hat{\mathcal{L}}_O^{J-1} W)|_{p,S} \leq C^{(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta + \underline{\Gamma}(\lambda))}}{\rho_{0,1}^J}, \quad (10.17)$$

the following estimate holds for the norms of $\mathcal{L}_O^{J-1} \Psi(W)$,

$$|\mathcal{L}_O^{J-1} \Psi|_{p,S} \leq C^{(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta + \underline{\Gamma}(\lambda))}}{\rho_{0,1}^J} \quad (10.18)$$

which completes the proof of Theorem 4.1.

Lemma 10.4 *Let us assume that for any J the estimates 10.18 hold, let us assume that the connection coefficients satisfy, in the internal region, with all their angular derivatives up to order $J-1$ the estimates 3.29, then the null components Ψ of the Riemann tensor satisfy the following estimates for $6 < N \leq J-1$ with $\rho < \rho_{0,1}$,*

$$|\lambda|_{\underline{\Gamma}(\Psi)}^{\phi(\Psi)} r^{\phi(\Psi) + (J-1) - \frac{2}{p}} \nabla^{J-1} \Psi|_{L^p(S)} \leq C^{(1)} e^{(J-2)\delta} e^{(J-2)\underline{\Gamma}(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho^J}, \quad (10.19)$$

⁹³This inequality is slightly stronger than the one proved in [Kl-Ni2] due to the fact that we are considering angular derivatives greater than 3 which implies that to prove the boundedness of the $\mathcal{Q}_{(1)}^{(J-1)}(C(\lambda; [\nu_0, \nu])) + \underline{\mathcal{Q}}_{(1)}^{(J-1)}(\underline{C}(\nu; [\lambda_0, \lambda]))$ norms we can use the inductive assumptions for the connection coefficients up to third order.

with $C^{(1)}$ of order $O(1)$ for $J \geq 7$, and $O(\varepsilon)$ for $J < 7$,⁹⁴ see [Kl-Ni2], chapter 6.

$$\begin{aligned} \phi(\alpha) &= \phi(\beta) = \frac{7}{2}, \quad \phi(\rho) = \phi(\sigma) = 3, \quad \phi(\underline{\beta}) = 2, \quad \phi(\underline{\alpha}) = 1 \\ \underline{\phi}(\alpha) &= \underline{\phi}(\beta) = 0, \quad \underline{\phi}(\rho) = \underline{\phi}(\sigma) = \frac{1}{2}, \quad \underline{\phi}(\underline{\beta}) = \frac{3}{2}, \quad \underline{\phi}(\underline{\alpha}) = \frac{5}{2}. \end{aligned} \quad (10.20)$$

Proof: See Appendix to section 10.

11 The complete results of Section 5

11.0.6 The Ω component

The angular derivatives of this component are easy to control as

$$\partial \log \Omega = \nabla \log \Omega = 2^{-1}(\eta + \underline{\eta})$$

and η and $\underline{\eta}$ are already under control. Therefore

$$\nabla^N \log \Omega = \frac{1}{2}(\nabla^{N-1} \eta + \nabla^{N-1} \underline{\eta}) \quad (11.1)$$

and

$$\begin{aligned} \nabla^N \Omega &= \nabla^N e^{\log \Omega} = \sum_{k=0}^{\infty} \frac{1}{k!} \nabla^N (\log \Omega)^k = \sum_{k=1}^{\infty} \frac{1}{k!} \nabla^N (\log \Omega)^k \\ &= N! \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\gamma_1, \dots, \gamma_k}^{\gamma_s=N; \gamma_s \in [0, N]} \frac{1}{\gamma_1! \gamma_2! \cdots \gamma_k!} (\nabla^{\gamma_1} \log \Omega) (\nabla^{\gamma_2} \log \Omega) \cdots (\nabla^{\gamma_k} \log \Omega) \end{aligned}$$

Therefore

$$\begin{aligned} |\nabla^N \Omega|_{p,S} &\leq N! \sum_{k=1}^{\infty} \frac{1}{k!} \left[\sum_{\gamma_1=0}^{\frac{N}{2}} \sum_{\gamma_2, \dots, \gamma_k}^{\gamma_s=N-\gamma_1; \gamma_s \in [0, N]} \frac{1}{\gamma_1! \gamma_2! \cdots \gamma_k!} |(\nabla^{\gamma_1} \log \Omega) (\nabla^{\gamma_2} \log \Omega) \cdots (\nabla^{\gamma_k} \log \Omega)|_{p,S} \right. \\ &\quad \left. \sum_{\gamma_1=\frac{N}{2}+1}^N \sum_{\gamma_2, \dots, \gamma_k}^{\gamma_s=N-\gamma_1; \gamma_s \in [0, N]} \frac{1}{\gamma_1! \gamma_2! \cdots \gamma_k!} |(\nabla^{\gamma_1} \log \Omega) (\nabla^{\gamma_2} \log \Omega) \cdots (\nabla^{\gamma_k} \log \Omega)|_{p,S} \right]. \end{aligned}$$

From it we obtain, see Section 17, the following estimate,

$$|\nabla^N \Omega|_{p,S} \leq \left(\tilde{C}_4 \frac{(N-1)!}{(N-1)^\alpha} \frac{e^{(N-3)(\delta+\mathbb{L}(\lambda))}}{\rho^{N-1}} \right). \quad (11.2)$$

⁹⁴If $J \geq 1$, for $J = 0$ ρ becomes $\rho - \underline{\rho}$

Adapting the same techniques used before to prove the estimates for the angular derivatives of the connection coefficients we prove the following estimate,

$$|\partial_\nu^N \Omega|_{p,S} \leq c \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta+\Gamma(\lambda))}}{\rho^N} \right) . \quad (11.3)$$

The detailed proof of 11.3 is in Section 17.

11.0.7 The γ components

We recall that γ_{ab} satisfies the following equation

$$\frac{\partial}{\partial \nu} \gamma_{ab} - \Omega \text{tr} \chi \gamma_{ab} = 2\Omega \hat{\chi}_{ab} , \quad (11.4)$$

which we rewrite as

$$\frac{\partial}{\partial \nu} \gamma_{ab} - \overline{\Omega \text{tr} \chi} \gamma_{ab} = (\Omega \text{tr} \chi - \overline{\Omega \text{tr} \chi}) \gamma_{ab} + 2\Omega \hat{\chi}_{ab} \quad (11.5)$$

and

$$\frac{\partial}{\partial \nu} (r^{-2} \gamma_{ab}) = (\Omega \text{tr} \chi - \overline{\Omega \text{tr} \chi}) (r^{-2} \gamma_{ab}) + 2\Omega r^{-2} \hat{\chi}_{ab} .$$

The control of the angular non covariant derivatives can be easily obtained applying $\not\partial^J$ to both sides. The following estimates hold

$$|r^{J-2-\frac{2}{p}} \not\partial^J \gamma_{ab}|_{p,S} \leq c \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho^J} \right) . \quad (11.6)$$

Remark:

The estimates for $\not\partial^J \gamma_{ab}$ follow with J instead of $J-1$ due to the loss of derivatives present in the transport equation 11.4. This loss is not present considering the mixed derivatives.

11.0.8 The X^a components

We study the $X_a = \gamma_{ac} X^c$ components which is equivalent; we have the transport equation

$$\frac{\partial X_a}{\partial \nu} = -4\Omega^2 \zeta_a . \quad (11.7)$$

Let us look at the transport equation for the norm $|X| = \sqrt{\gamma_{ab} X^a X^b}$,

$$\begin{aligned} \frac{\partial |X|^2}{\partial \nu} &= \frac{\partial \gamma_{ab}}{\partial \nu} X^a X^b + 2\gamma_{ab} X^a \frac{\partial X^b}{\partial \nu} = \Omega \text{tr} \chi |X|^2 + 2\Omega X \cdot \hat{\chi} \cdot X + 2\gamma_{ab} X^a \frac{\partial \gamma^{bc} X_c}{\partial \nu} \\ &= \Omega \text{tr} \chi |X|^2 + 2\Omega X \cdot \hat{\chi} \cdot X + 2\gamma_{ab} X^a \frac{\partial \gamma^{bc}}{\partial \nu} X_c + 2\gamma_{ab} \gamma^{bc} X^a \frac{\partial X_c}{\partial \nu} \\ &= \Omega \text{tr} \chi |X|^2 + 2\Omega X \cdot \hat{\chi} \cdot X - 2 \frac{\partial \gamma_{ab}}{\partial \nu} X^a X^b + 2X^c \zeta_c \\ &= \Omega \text{tr} \chi |X|^2 + 2\Omega X \cdot \hat{\chi} \cdot X - 2\Omega \text{tr} \chi |X|^2 - 4\Omega X \cdot \hat{\chi} \cdot X + 2X \cdot \tilde{\zeta} \\ &= -\Omega \text{tr} \chi |X|^2 - 2\Omega X \cdot \hat{\chi} \cdot X + 2X \cdot \tilde{\zeta} , \end{aligned} \quad (11.8)$$

where $\tilde{\zeta} = -4\Omega^2\zeta$. Therefore

$$\frac{\partial|X|^2}{\partial\nu} + \Omega\text{tr}\chi|X|^2 = -2\Omega X \cdot \hat{\chi} \cdot X + 2X \cdot \tilde{\zeta} \quad (11.9)$$

To obtain the transport equation for $\nabla^J|X|^2$ we proceed in the following way,

$$\nabla^J \frac{\mathcal{D}}{\partial\nu} |X|^2 = \frac{\mathcal{D}}{\partial\nu} (\nabla^J |X|^2) + [\nabla^J, \frac{\mathcal{D}}{\partial\nu}] |X|^2 \quad (11.10)$$

and

$$\begin{aligned} \frac{\mathcal{D}}{\partial\nu} (\nabla^J |X|^2) &= \nabla^J \frac{\mathcal{D}}{\partial\nu} |X|^2 - [\nabla^J, \frac{\mathcal{D}}{\partial\nu}] |X|^2 \\ &= \nabla^J \frac{\mathcal{D}}{\partial\nu} |X|^2 - \left[\sum_{k=0}^{J-1} \binom{J}{k} (\nabla^{J-1-k} \Omega\chi) \cdot \nabla^{k+1} |X|^2 - \sum_{k=0}^{J-1} \binom{J+1}{k} (\nabla^{J-1-k} C) \nabla^k |X|^2 \right] \\ &= -J \frac{\Omega\text{tr}\chi}{2} \cdot \nabla^J |X|^2 - J(\Omega\hat{\chi}) \cdot \nabla^J |X|^2 + \nabla^J \frac{\mathcal{D}}{\partial\nu} |X|^2 \\ &\quad - \left[\sum_{k=0}^{J-2} \binom{J}{k} (\nabla^{J-1-k} \Omega\chi) \cdot \nabla^{k+1} |X|^2 - \sum_{k=0}^{J-1} \binom{J+1}{k} (\nabla^{J-1-k} C) \nabla^k |X|^2 \right] \end{aligned} \quad (11.11)$$

As

$$\begin{aligned} \nabla^J \frac{\mathcal{D}}{\partial\nu} |X|^2 &= -\nabla^J [\Omega\text{tr}\chi|X|^2 + 2\Omega X \cdot \hat{\chi} \cdot X - 2X \cdot \zeta] \\ &= -\sum_{k=0}^J \binom{J}{k} (\nabla^k \Omega\text{tr}\chi) \nabla^{J-k} |X|^2 - \sum_{k=0}^J \sum_{l=0}^{J-k} \frac{J!}{k!l!(J-k-l)!} (\nabla^k X) (\nabla^l \Omega\hat{\chi}) (\nabla^{J-k-l} X) \\ &\quad + 2 \sum_{k=0}^J \sum_{l=0}^{J-k} \frac{J!}{k!(J-k)!} (\nabla^k X) \nabla^{J-k} \zeta \\ &= -\Omega\text{tr}\chi (\nabla^J |X|^2) - 2(\nabla^J X) \cdot \hat{\chi} \cdot X + 2(\nabla^J X) \cdot \tilde{\zeta} \\ &\quad - \left[\sum_{k=1}^J \binom{J}{k} (\nabla^k \Omega\text{tr}\chi) \nabla^{J-k} |X|^2 + \sum_{\substack{s=1 \\ \gamma_1 \gamma_2 \gamma_3}}^4 \frac{J!}{\gamma_1! \gamma_2! \gamma_3!} (\nabla^{\gamma_1} X) \cdot (\nabla^{\gamma_2} \Omega\hat{\chi}) \cdot (\nabla^{\gamma_3} X) \right. \\ &\quad \left. + 2 \sum_{k=0}^{J-1} \sum_{l=0}^{J-k} \frac{J!}{k!(J-k)!} (\nabla^k X) \nabla^{J-k} \tilde{\zeta} \right] \end{aligned} \quad (11.12)$$

substituting in the previous expression we have

$$\frac{\mathcal{D}}{\partial\nu} (\nabla^J |X|^2) + (J+2) \frac{\Omega\text{tr}\chi}{2} \cdot \nabla^J |X|^2 + J(\Omega\hat{\chi}) \cdot \nabla^J |X|^2 - 2(\nabla^J X) \cdot \hat{\chi} \cdot X + 2(\nabla^J X) \cdot \tilde{\zeta}$$

$$\begin{aligned}
&= - \left\{ \left[\sum_{k=0}^{J-2} \binom{J}{k} (\nabla^{J-1-k} \Omega_\chi) \cdot \nabla^{k+1} |X|^2 - \sum_{k=0}^{J-1} \binom{J+1}{k} (\nabla^{J-1-k} C) \nabla^k |X|^2 \right] \right. \\
&\quad - \left[\sum_{k=1}^J \binom{J}{k} (\nabla^k \Omega_{\text{tr} \chi}) \nabla^{J-k} |X|^2 + \sum_{\substack{s=1 \\ \gamma_1 \gamma_2 \gamma_3}}^4 \gamma_s = J; \gamma_1, \gamma_3 \neq J \frac{J!}{\gamma_1! \gamma_2! \gamma_3!} (\nabla^{\gamma_1} X) \cdot (\nabla^{\gamma_2} \Omega_{\hat{\chi}}) \cdot (\nabla^{\gamma_3} X) \right. \\
&\quad \left. \left. + 2 \sum_{k=0}^{J-1} \sum_{l=0}^{J-k} \frac{J!}{k!(J-k)!} (\nabla^k X) \nabla^{J-k} \tilde{\zeta} \right] \right\} \quad (11.13)
\end{aligned}$$

which we rewrite in a more compact way as

$$\frac{\mathcal{D}}{\partial \nu} (\nabla^J |X|^2) + (J+2) \frac{\Omega_{\text{tr} \chi}}{2} \cdot \nabla^J |X|^2 + J(\Omega_{\hat{\chi}}) \cdot \nabla^J |X|^2 - 2(\nabla^J X) \cdot \Omega_{\hat{\chi}} \cdot X + 2(\nabla^J X) \cdot \tilde{\zeta} = \left\{ \text{good} \right\}_X \quad (11.14)$$

where

$$\begin{aligned}
\left\{ \text{good} \right\}_X &= - \left\{ \left[\sum_{k=0}^{J-2} \binom{J}{k} (\nabla^{J-1-k} \Omega_\chi) \cdot \nabla^{k+1} |X|^2 - \sum_{k=0}^{J-1} \binom{J+1}{k} (\nabla^{J-1-k} C) \nabla^k |X|^2 \right] \right. \\
&\quad - \left[\sum_{k=1}^J \binom{J}{k} (\nabla^k \Omega_{\text{tr} \chi}) \nabla^{J-k} |X|^2 + \sum_{\substack{s=1 \\ \gamma_1 \gamma_2 \gamma_3}}^4 \gamma_s = J; \gamma_1, \gamma_3 \neq J \frac{J!}{\gamma_1! \gamma_2! \gamma_3!} (\nabla^{\gamma_1} X) \cdot (\nabla^{\gamma_2} \Omega_{\hat{\chi}}) \cdot (\nabla^{\gamma_3} X) \right. \\
&\quad \left. \left. + 2 \sum_{k=0}^{J-1} \sum_{l=0}^{J-k} \frac{J!}{k!(J-k)!} (\nabla^k X) \nabla^{J-k} \tilde{\zeta} \right] \right\}.
\end{aligned}$$

From this equation one gets the transport equation for the $|\cdot|_{p,S}$ norms with the appropriate weights and proceeding as done many times before, we prove the following estimate for all J ,

$$\left| \frac{r^{(J+1-\frac{2}{p})}}{\log r} \nabla^J |X| \right|_{p,S}(\lambda, \nu) \leq c\varepsilon_0 \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\Gamma_1+\delta)}}{\rho^J} \right). \quad (11.15)$$

12 The complete results of Section 6

12.1 The control of the (non covariant) partial derivatives

Assume for simplicity \mathcal{O} denotes a S -tangent vector, for instance the connection coefficient ζ ; we are interested to the analyticity of the tensor field $\zeta = \zeta(e_C) \theta^C(\cdot)$ which means that we have to prove that the various components $\zeta(e_C)(\lambda, \nu, \omega^a)$ are analytic functions in the λ, ν, ω^a variables. To prove it we have to control the norm of the derivatives of these components. As discussed in (I) this implies that we have to control the (norms of the) mixed derivatives in ν, ω^a for the quantities defined on an outgoing cone and the mixed derivatives in λ, ω^a for the quantities defined on an incoming cone. We start looking at the first situation observing that the second case can be treated exactly in the same way.

12.1.1 The non covariant mixed derivatives in $\{\nu, \omega^a\}$.

Preliminary we recall some trivial facts about the covariant derivatives, let V be a covariant vector field,

$$\begin{aligned}
|\nabla V|^2 &= \nabla_\mu V_\nu \nabla_\rho V_\sigma g^{\mu\rho} g^{\nu\sigma} = \sum_{A,B} \nabla_\mu V_\nu \nabla_\rho V_\sigma e_A^\mu e_A^\rho e_B^\nu e_B^\sigma \\
&= \sum_{A,B} (\nabla_A V)_\nu e_B^\nu (\nabla_A V)_\sigma e_B^\sigma = \sum_{A,B} ((\partial_A V)_\nu e_B^\nu - \Gamma_{AB}^\nu V_\nu) ((\partial_A V)_\sigma e_B^\sigma - \Gamma_{AB}^\sigma V_\sigma) \\
&= \sum_{A,B} ((\partial_A V)_\nu e_B^\nu - \Gamma_{AB}^\nu V_\nu)^2 .
\end{aligned} \tag{12.1}$$

Observe that

$$((\partial_A V)_\nu e_B^\nu - \Gamma_{AB}^\nu V_\nu) = \partial_A(V(e_B)) - (\partial_A e_B)^\nu V_\nu - \Gamma_{AB}^\nu V_\nu$$

and

$$(\partial_A e_B)^\nu \neq \Gamma_{AB}^\nu .$$

We have the following relations,

$$\begin{aligned}
\partial_{e_A} \zeta(e_B) &= (\mathbf{D}_{e_A} \zeta)(e_B) + \zeta(e_C) \mathbf{g}(e_C, \mathbf{D}_{e_A} e_B) \\
&= (\nabla_{e_A} \zeta)(e_B) - \zeta(e_C) \mathbf{g}(\nabla_{e_A} e_C, e_B)
\end{aligned}$$

$$\begin{aligned}
\partial_\nu \zeta(e_B) &= (\Omega \mathbf{D}_{e_4} \zeta)(e_B) + \Omega \zeta(e_C) \mathbf{g}(e_C, \mathbf{D}_{e_4} e_B) \\
&= (\mathfrak{D}_\nu \zeta)(e_B) - \zeta(e_C) \Omega \mathbf{g}(\mathbf{D}_{e_4} e_C, e_B) \\
&= (\mathfrak{D}_\nu \zeta)(e_B) - \zeta(e_C) \Omega \mathbf{g}(\mathbf{D}_{e_C} e_4, e_B) - \zeta(e_C) (\cdot) \Omega \mathbf{g}([e_C, e_4], e_B) \\
&= (\mathfrak{D}_\nu \zeta)(e_B) - \Omega \zeta(e_C) \chi(e_C, e_B) - \Omega \zeta(e_C) \mathbf{g}([e_C, e_4], e_B) \\
&= (\mathfrak{D}_\nu \zeta)(e_B) - 2\Omega \zeta(e_C) \chi(e_C, e_B)
\end{aligned}$$

where we used the relation

$$[e_C, e_4] = -\mathfrak{D}_4 e_C + \chi(e_C, e_D) e_D - \nabla_A (\log \Omega) e_4 \tag{12.2}$$

and assumed we have chosen a Fermi transported frame, implying $\mathfrak{D}_4 e_C = 0$. Therefore we can write

$$\partial_\alpha \zeta = \nabla_\alpha \zeta - \tilde{c}_\alpha \cdot \zeta = (\nabla_\alpha - \tilde{c}_\alpha \cdot) \zeta \tag{12.3}$$

where

$$\begin{aligned}
\nabla_A &= \nabla_{e_A} , \quad \nabla_\nu = \Omega \mathfrak{D}_4 \\
\tilde{c}_A &= -\mathbf{g}(\nabla_{e_A} e_C, e_B) (\cdot) , \quad \tilde{c}_\nu = 2\Omega \chi .
\end{aligned} \tag{12.4}$$

Therefore we write in a symbolic way,

$$\partial \mathcal{O} = (\nabla - c) \mathcal{O} \tag{12.5}$$

and, symbolically,

$$\partial^{J-1}\mathcal{O} = (\nabla - c)^{J-1}\mathcal{O} \text{ “=” } \sum_{k=0}^{J-1} \binom{J-1}{k} \nabla^k c^{J-1-k} \mathcal{O} \quad (12.6)$$

where in the right hand side we have to be careful at the position of the ∇ 's as they can operate on the c 's or on the \mathcal{O} . Therefore fixed k one has to interpret the right hand side as made by $J-k$ slots, each one at the left of one of the c 's we think as numbered and the last one immediately at the left of the \mathcal{O} . In these $J-k$ slots one has to distribute k ∇ 's, each one operating exclusively on the tensor c at its immediate right, or on \mathcal{O} if we are considering the last slot. Computing all the possible ways of distributing these ∇ 's one is considering more terms of the real existing ones, but this over estimate should not be harmful. Therefore we write, omitting the minus signs as at the end we are doing a norm estimate, the following expression⁹⁵

$$\partial^{J-1}\mathcal{O} \text{ “=” } \sum_{k=0}^{J-1} \binom{J-1}{k} \sum_{\gamma_1, \gamma_2, \dots, \gamma_{J-k}}^{\sum_{s=1}^{J-k} \gamma_s = k} \frac{k!}{\gamma_1! \gamma_2! \dots \gamma_{J-k}!} (\nabla^{\gamma_1} c) (\nabla^{\gamma_2} c) \dots (\nabla^{\gamma_{J-1-k}} c) (\nabla^{\gamma_{J-k}} \mathcal{O})$$

and its norm estimate is, omitting for notation simplicity the r weights and using the estimates proved in Theorem 9.8

$$|\partial^{J-1}\mathcal{O}|_{p,S} \leq \sum_{k=0}^{J-1} \binom{J-1}{k} \sum_{\gamma_1, \gamma_2, \dots, \gamma_{J-k}}^{\sum_{s=1}^{J-k} \gamma_s = k} \frac{k!}{\gamma_1! \gamma_2! \dots \gamma_{J-k}!} |\nabla^{\gamma_1} c|_{\infty,S} |\nabla^{\gamma_2} c|_{\infty,S} \dots |\nabla^{\gamma_{J-1-k}} c|_{\infty,S} |\nabla^{\gamma_{J-k}} \mathcal{O}|_{p,S}$$

which we rewrite, denoting $q = \gamma_{J-k}$,

$$\begin{aligned} |\partial^{J-1}\mathcal{O}|_{p,S} &\leq \sum_{k=0}^{J-1} \sum_{q=0}^k \binom{J-1}{k} \binom{k}{q} |\nabla^q \mathcal{O}|_{p,S} \sum_{\gamma_1, \gamma_2, \dots, \gamma_{J-1-k}}^{\sum_{s=1}^{J-1-k} \gamma_s = k-q} \frac{(k-q)!}{\gamma_1! \gamma_2! \dots \gamma_{J-1-k}!} |\nabla^{\gamma_1} c|_{\infty,S} |\nabla^{\gamma_2} c|_{\infty,S} \dots |\nabla^{\gamma_{J-1-k}} c|_{\infty,S} \\ &\leq (J-1)! \sum_{k=0}^{J-1} \frac{1}{(J-1-k)!} \sum_{q=0}^k \frac{|\nabla^q \mathcal{O}|_{p,S}}{q!} \sum_{\gamma_1, \gamma_2, \dots, \gamma_{J-1-k}}^{\sum_{s=1}^{J-1-k} \gamma_s = k-q} \frac{1}{\gamma_1! \gamma_2! \dots \gamma_{J-1-k}!} |\nabla^{\gamma_1} c|_{\infty,S} |\nabla^{\gamma_2} c|_{\infty,S} \dots |\nabla^{\gamma_{J-1-k}} c|_{\infty,S} \\ &\leq \left(c \frac{(J-1)!}{(J-1)^\alpha} \right) \left\{ \sum_{k=0}^{J-1} \frac{1}{(J-1-k)!} \sum_{q=0}^k \frac{(J-1)^\alpha}{(\epsilon(q) + q)^\alpha} \frac{e^{(q-2)(\delta_0 + \Gamma)}}{\rho^q} \right. \\ &\quad \cdot \left. \left[\sum_{\gamma_1, \gamma_2, \dots, \gamma_{J-1-k}}^{\sum_{s=1}^{J-1-k} \gamma_s = k-q} \frac{1}{\gamma_1! \gamma_2! \dots \gamma_{J-1-k}!} \frac{(\gamma_1 + 1)!}{(\gamma_1 + 1)^\alpha} \frac{e^{(\gamma_1 - 1)(\delta + \Gamma(\lambda))}}{\rho^{\gamma_1 + 1}} \frac{(\gamma_2 + 1)!}{(\gamma_2 + 1)^\alpha} \frac{e^{(\gamma_2 - 1)(\delta + \Gamma(\lambda))}}{\rho^{\gamma_2 + 1}} \dots \frac{(\gamma_{J-1-k} + 1)!}{(\gamma_{J-1-k} + 1)^\alpha} \frac{e^{(\gamma_{J-1-k} - 1)(\delta + \Gamma(\lambda))}}{\rho^{\gamma_{J-1-k} + 1}} \right] \right\} \end{aligned} \quad (12.7)$$

⁹⁵Recall that the value of ρ has been lowered to have also for the mixed derivatives an estimate with $\frac{(J+P)!}{(J+P)^\alpha}$ instead of $\frac{(J+P+1)!}{(J+P+1)^\alpha}$.

where $\epsilon(0) = 1, \epsilon(q) = 0, q > 0$. We write

$$\begin{aligned}
|\partial^{J-1}\mathcal{O}|_{p,S} &\leq c \left(\frac{(J-1)!}{(J-1)^\alpha} \right) \cdot \left(\frac{e^{[(q-2)+(k-q)-(J-1-k)](\delta+\Gamma(\lambda))}}{\rho^{q+(k-q)+J-1-k}} \right) \left\{ \sum_{k=0}^{J-1} \frac{1}{(J-1-k)!} \sum_{q=0}^k \frac{(J-1)^\alpha}{(\epsilon(q)+q)^\alpha} \right. \\
&\quad \cdot \left. \left[\sum_{s=1}^{J-1-k} \sum_{\gamma_1, \gamma_2, \dots, \gamma_{J-1-k}}^{\gamma_s=k-q} \frac{1}{\gamma_1! \gamma_2! \dots \gamma_{J-1-k}!} \frac{(\gamma_1+1)!}{(\gamma_1+1)^\alpha} \frac{(\gamma_2+1)!}{(\gamma_2+1)^\alpha} \dots \frac{(\gamma_{J-1-k}+1)!}{(\gamma_{J-1-k}+1)^\alpha} \right] \right\} \\
&\leq c \left(\frac{(J-1)!}{(J-1)^\alpha} \frac{e^{((J-1)-2)(\delta+\Gamma(\lambda))}}{\rho^{J-1}} \right) \left\{ \sum_{k=0}^{J-1} \frac{1}{(J-1-k)!} \sum_{q=0}^k \frac{(J-1)^\alpha}{(\epsilon(q)+q)^\alpha} \cdot \left[\sum_{s=1}^{J-1-k} \sum_{\gamma_1, \gamma_2, \dots, \gamma_{J-1-k}}^{\gamma_s=k-q} \frac{1}{\gamma_1^{\alpha-1} \gamma_2^{\alpha-1} \dots \gamma_{J-1-k}^{\alpha-1}} \right] \right\} \\
&\leq c \left(\frac{(J-1)!}{(J-1)^\alpha} \frac{e^{((J-1)-2)(\delta+\Gamma(\lambda))}}{\rho^{J-1}} \right) \left\{ \sum_{k=0}^{J-1} \frac{c_1^{J-1-k}}{(J-1-k)!} \sum_{q=0}^k \frac{(J-1)^\alpha}{(\epsilon(q)+q)^\alpha} \right\}.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\left\{ \sum_{k=0}^{J-1} \frac{c_1^{J-1-k}}{(J-1-k)!} \sum_{q=0}^k \frac{(J-1)^\alpha}{(\epsilon(q)+q)^\alpha} \right\} \tag{12.8} \\
&= \left\{ \sum_{k=0}^{\lfloor \frac{J-1}{2} \rfloor} \frac{c_1^{J-1-k}}{(J-1-k)!} \sum_{q=0}^k \frac{(J-1)^\alpha}{(\epsilon(q)+q)^\alpha} \right\} + \left\{ \sum_{k=\lfloor \frac{J-1}{2} \rfloor + 1}^{J-1} \frac{c_1^{J-1-k}}{(J-1-k)!} \sum_{q=0}^k \frac{(J-1)^\alpha}{(\epsilon(q)+q)^\alpha} \right\} \\
&\leq c \left\{ \sum_{k=0}^{\lfloor \frac{J-1}{2} \rfloor} \frac{c_1^{J-1-k}}{(J-1-k)!} \sum_{q=0}^k \frac{1}{(\epsilon(q)+q)^\alpha} \right\} + \left\{ \sum_{k=\lfloor \frac{J-1}{2} \rfloor + 1}^{J-1} \frac{c_1^{J-1-k}}{(J-1-k)!} \sum_{q=0}^k \frac{(J-1)^\alpha}{(\epsilon(q)+q)^\alpha} \right\}
\end{aligned}$$

and, choosing $\hat{\rho}/\rho$ sufficiently small, it follows that

$$|\partial^{J-1}\mathcal{O}|_{p,S} \leq c \left(\frac{(J-1)!}{(J-1)^\alpha} \frac{e^{((J-1)-2)(\delta+\Gamma(\lambda))}}{\hat{\rho}^{J-1}} \right). \tag{12.9}$$

12.2 The $O^{(i)}$ components

In the estimates for the null Riemann components, see the appendix to Section 10, we use repeatedly the estimates of the tangential derivatives of the $O^{(i)}$ components. To prove these estimates we proceed in the following way; let us considered first the simplified case where the $O^{(i)}$ have the same expression as in the Minkowski spacetime, therefore

$$\begin{aligned}
O^{(1)} &= -r \sin \phi e_\theta - r \cos \phi \cos \theta e_\phi \\
O^{(2)} &= r \cos \phi e_\theta - r \sin \phi \cos \theta e_\phi \\
O^{(3)} &= r \sin \theta e_\phi
\end{aligned} \tag{12.10}$$

and summarizing

$${}^{(i)}O^c = \sum_C f_C^{(i)}(r, \theta, \phi) e_C^c \quad (12.11)$$

where $f_C^{(i)}(r, \theta, \phi)$ are rational functions in r, θ, ϕ . let us consider the norm of $\nabla^{(i)}O$,

$$\begin{aligned} |\nabla^{(i)}O|^2 &= \nabla_a {}^{(i)}O^b \nabla_{\bar{a}} {}^{(i)}O^{\bar{b}} \gamma^{a\bar{a}} \gamma_{b\bar{b}} = \nabla_a {}^{(i)}O^b \nabla_{\bar{a}} {}^{(i)}O^{\bar{b}} \sum_C e_C^a e_{\bar{C}}^{\bar{a}} \sum_D \theta_b^D \theta_{\bar{b}}^{\bar{D}} \\ &= \sum_C \sum_D \nabla_C {}^{(i)}O^b \nabla_C {}^{(i)}O^{\bar{b}} \theta_b^D \theta_{\bar{b}}^{\bar{D}} = \sum_C \sum_D \mathbf{g}(e_D, \nabla_C {}^{(i)}O) \mathbf{g}(e_D, \nabla_C {}^{(i)}O) \\ &= \sum_D \mathbf{g}(e_D, \nabla^{(i)}O) \mathbf{g}(e_D, \nabla^{(i)}O) = \sum_D |\mathbf{g}(e_D, \nabla^{(i)}O)|^2 \\ &= \sum_D \left| \sum_C \mathbf{g}(e_D, \nabla f_C^{(i)} e_C) \right|^2 = \sum_D \left| \sum_C \nabla f_C^{(i)} \mathbf{g}(e_D, e_C) + \sum_C f_C^{(i)} \mathbf{g}(e_D, \nabla e_C) \right|^2 \\ &\leq 2 \sum_D |\nabla f_D^{(i)}|^2 + 2 \sum_D \sum_C |f_C^{(i)}|^2 |\mathbf{g}(e_D, \nabla e_C)|^2 \leq 2^2 |\nabla f^{(i)}|^2 + 2^3 |f^{(i)}|^2 |\mathbf{g}(e., \nabla e.)|^2 \end{aligned} \quad (12.12)$$

where

$$\begin{aligned} |\mathbf{g}(e., \nabla e.)|^2 &= \sup_{C,D} |\mathbf{g}(e_D, \nabla e_C)|^2 \\ |\nabla f^{(i)}|^2 &= \sup_C |\nabla f_C^{(i)}|^2, \quad |f^{(i)}|^2 = \sup_C |f_C^{(i)}|^2. \end{aligned} \quad (12.13)$$

The estimate for $|\nabla f^{(i)}|^2$ and for $|f^{(i)}|^2$ are better than those inductively assumed,⁹⁶ therefore we are left only with the estimate of $|\mathbf{g}(e_D, \nabla e_C)|^2$ obtained in subsection 9.2. More in general we have

$$\nabla^J {}^{(i)}O = \sum_C \nabla^J f_C^{(i)}(r, \theta, \phi) e_C = \sum_C \sum_{k=0}^J \binom{J}{k} (\nabla^k f_C^{(i)}(r, \theta, \phi)) \nabla^{J-k} e_C \quad (12.14)$$

and, for a generic norm,

$$\begin{aligned} |\nabla^J {}^{(i)}O| &\leq \sum_C \sum_{k=0}^J \binom{J}{k} |\nabla^k f_C^{(i)}| |\nabla^{J-k} e_C| \leq \sum_C \sum_{k=0}^J \binom{J}{k} |\nabla^k f_C^{(i)}| \sqrt{\sum_D \mathbf{g}(e_D, \nabla^{J-k} e_C)^2} \\ &\leq \sum_C \sum_D \sum_{k=0}^J \binom{J}{k} |\nabla^k f_C^{(i)}| |\mathbf{g}(e_D, \nabla^{J-k} e_C)|. \end{aligned} \quad (12.15)$$

Therefore

$$|\nabla^J {}^{(i)}O|_{p,S} \leq \sum_C \sum_D \sum_{k=0}^J \binom{J}{k} |\nabla^k f_C^{(i)}|_{\infty,S} |\mathbf{g}(e_D, \nabla^{J-k} e_C)|_{p,S}$$

⁹⁶ Apart from corrections at the level of metric components.

$$\leq 2^2 \sum_{k=0}^J \binom{J}{k} |\nabla^k f^{(i)}|_{\infty, S} \left(\sup_{C, D} |\mathbf{g}(e_D, \nabla^{J-k} e_C)|_{p, S} \right) . \quad (12.16)$$

As we already have the control of the $|\mathbf{g}(e_D, \nabla^{J-k} e_C)|_{p, S}$ norms, recalling that these norms are different from zero even in the Minkowski case, the final result is

$$|\nabla^{J(i)} O|_{p, S} \leq c \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta + \Gamma(\lambda))}}{\rho^J} \right) . \quad (12.17)$$

13 The complete results of Section 7

13.1 The initial data on the outgoing cone

Let us write the transport equations which the initial data have to satisfy along C_0 , see equations (I;2.28), (I;2.30),

$$\frac{\partial}{\partial \nu} \gamma_{ab} = 2\Omega \chi_{ab} \quad , \quad \frac{\partial}{\partial \nu} \log \Omega = -2\Omega \omega$$

$$\mathbf{D}_4 \text{tr} \chi + \frac{1}{2} (\text{tr} \chi)^2 + 2\omega \text{tr} \chi + |\hat{\chi}|^2 = 0$$

$$\mathcal{D}_4 \zeta + \zeta \chi + \text{tr} \chi \zeta - \text{div} \chi + \nabla \text{tr} \chi + \mathcal{D}_4 \nabla \log \Omega = 0$$

$$\mathbf{D}_4 \text{tr} \underline{\chi} + \text{tr} \chi \text{tr} \underline{\chi} - 2\omega \text{tr} \underline{\chi} + 2 \text{div} (\zeta - \nabla \log \Omega) - 2|\zeta - \nabla \log \Omega|^2 = -2\mathbf{K}$$

$$\mathcal{D}_4 \hat{\chi} + \frac{1}{2} \text{tr} \chi \hat{\chi} + \frac{1}{2} \text{tr} \underline{\chi} \hat{\chi} - 2\omega \hat{\chi} + \nabla \hat{\chi} (\zeta - \nabla \log \Omega) - (\zeta - \nabla \log \Omega) \hat{\chi} (\zeta - \nabla \log \Omega) = 0$$

$$\mathbf{D}_4 \underline{\omega} - 2\omega \underline{\omega} - \zeta \cdot \nabla \log \Omega - \frac{3}{2} |\zeta|^2 + \frac{1}{2} |\nabla \log \Omega|^2 + \frac{1}{2} (\mathbf{K} + \frac{1}{4} \text{tr} \chi \text{tr} \underline{\chi} - \frac{1}{2} \hat{\chi} \cdot \hat{\chi}) = 0 .$$

On C_0 we assign freely Ω and $\hat{\chi}$. We require them to be analytic functions on the whole C_0 satisfying, for any $J > 0$, the following bounds, see also inequalities 3.31,

$$\begin{aligned} |r^{J+2+\epsilon-\frac{2}{p}} \nabla^J \log \Omega|_{p, S} &\leq C_3^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,0,1}^J} \\ |r^{J+\frac{5}{2}+\epsilon-\frac{2}{p}} \nabla^J \hat{\chi}|_{p, S} &\leq C_1^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,0,1}^J} . \end{aligned} \quad (13.1)$$

The strategy to prove the existence of the analytic initial data on C_0 is the following one:

i) We assume that on S_0 is assigned as a real analytic function satisfying a p.d.e. equation in the angular variables we describe later on and such that, for any $J > 0$, satisfies the following norm bound, where $U = \Omega^{-1} \text{tr} \chi$, with $\rho_{0,0} < \rho_{0,0,1}$,

$$|r^{J+2-\frac{2}{p}} \nabla^J U|_{p, S_0} \leq C_0^{(0,0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,0}^J} . \quad (13.2)$$

ii) We prove a simplified version of Theorem 9.1, namely we assume inequality 13.2 and on C_0 we make the following inductive assumptions, with $J < N$, for U and for ζ and prove the following theorem,

Theorem 13.1 *Assume that on S_0 is assigned as a real analytic function satisfying for any J the bound,*

$$|r^{J+2-\frac{2}{p}} \nabla^J U|_{p,S_0} \leq C_0^{(0,0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,0}^J}. \quad (13.3)$$

Assume that on C_0 , U and ζ satisfy the following initial conditions with $J < N$

$$|r^{J+2-\frac{2}{p}} \nabla^J U|_{p,S}(\lambda_0, \nu) \leq C_0^{(0)} (J-2)(\delta_0 + \Gamma_0(\lambda)) \frac{J!}{J^\alpha} \frac{1}{\rho_{0,0,1}^J} \quad (13.4)$$

$$|r^{J+2-\frac{2}{p}} \nabla^J \zeta|_{p,S}(\lambda_0, \nu) \leq C_4^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{(J)!}{(J)^\alpha} \frac{1}{\rho_{0,0,1}^J}$$

then $\nabla^N U(\lambda_0, \nu)$ satisfies the following estimate

$$|r^{N+2-\frac{2}{p}} \nabla^N U|_{p,S}(\lambda_0, \nu) \leq C_0^{(0)} e^{(J-2)(\delta_0 + \Gamma_0(\lambda))} \frac{N!}{N^\alpha} \frac{1}{\rho_{0,0,1}^N} \quad (13.5)$$

for any ν , with $\rho_{0,0,1} < \rho_{0,0}$, provided that the initial data are small.

Proof: The proof of this theorem mimicks the proof of Theorem 9.1, but is significantly simpler for the following reasons:

a) In Theorem 9.1 to control $\nabla^N U$ we have to look at the transport equation for $\nabla^N \psi$ to avoid when integrated, logarithmic divergent contributions. here due to the better decay assumed in 13.1 for the initial data, we can directly use the transport equation for $\nabla^N U$ as we show explicitly in the following remark. Moreover, looking at that theorem, we needed to express the $\nabla^N \hat{\chi}$ norms in terms of the $\nabla^N U$ norms. This required the use of the Hodge equation for $\hat{\chi}$ and a control of the $\nabla^{N-1} \beta$ norm. Here this is avoided as $\hat{\chi}$ is assigned freely on the whole C_0 and $\nabla^{N-1} \beta$ does not appear in the transport equation for $\nabla^N U$. Finally we do not have, in the present case, to integrate from the last slice.

b) In the control of the terms denoted collectively by $\{(good)_1\}$, see equation 13.8, contribution due to Riemann components appear, with a lower derivative order; due to a signature argument the only contribution is due to $\nabla^J \beta$ with $J \leq N-2$; again we do not need any hyperbolicity argument as we can use the relation,

$$\nabla^J \beta = \nabla^J [\nabla \text{tr} \chi - \text{div} \chi - \zeta \cdot \chi + \zeta \text{tr} \chi]$$

and control this term using the previous assumptions, 3.31, for U and ζ and the knowledge of $\hat{\chi}$ and its angular derivatives. Moreover in the estimates of lower order terms even $\nabla^J \Omega$ factors are present, which, again, can be controlled easily as Ω is assigned freely on $C_0 \cup \underline{C}_0$.

c) All this guarantees that the proof of Theorem 13.1 follows in a simpler way from the proof of Theorem 9.1. Still one step has to be done. Observe, in fact, that this inductive result requires a starting point namely that for $J = 1$ we have separately proved the estimate, this issue is delicate and has been proved in [Ca-Nil], see also subsection 13.1.1⁹⁷

$$|r^{1+2-\frac{2}{p}} \nabla U|_{p,S}(\lambda_0, \nu) \leq C_0^{(0)} e^{(1-2)(\delta_0 + \Gamma_0(\lambda))} \frac{1}{\rho_{0,0,1}}. \quad (13.6)$$

Remark:

As said in a) we prove Theorem 13.1 in a different way from the proof of Theorem 9.1 which presents some advantages when we prove the norms of $\nabla^J \zeta$ on C_0 . This is based on the fact that the r decay required⁹⁸ in the initial conditions 13.1 is, due to $\epsilon > 0$, stronger than the one proved in the internal region. This allows to control directly the ∇ derivatives of $U = \Omega^{-1} \text{tr} \chi$ instead the to obtain it through $\Psi = \nabla \Omega^{-1} \text{tr} \chi + \Omega^{-1} \text{tr} \chi \eta$. In fact the transport equation for $\nabla^J U$ is,

$$\frac{\mathcal{D}(\nabla^N U)}{\partial \nu} + \Omega((N+2) \frac{\text{tr} \chi}{2} + N \hat{\chi}) \cdot (\nabla^N U) + 2 \hat{\chi} \cdot (\nabla^N \hat{\chi}) = - \left\{ (good)_1 \right\} \quad (13.7)$$

where

$$\begin{aligned} \left\{ (good)_1 \right\} = & \left\{ \sum_{k=1}^{N-1} \binom{N}{k} (\nabla^k \hat{\chi}) \cdot (\nabla^{N-k} \hat{\chi}) - \frac{U}{2} \sum_{k=1}^N \binom{N}{k} (\nabla^k \Omega^2) (\nabla^{N-k} U) \right. \\ & \left. + \frac{1}{2} \sum_{k=1}^{N-1} \binom{N}{k} (\nabla^k \Omega \text{tr} \chi) \cdot (\nabla^{N-k} U) + \sum_{k=1}^{N-1} \binom{N}{k+1} (\nabla^k \Omega \chi) \cdot (\nabla^{N-k} U) - \sum_{k=1}^{N-1} \binom{N}{k+1} (\nabla^{k-1} C) \cdot (\nabla^{N-k} U) \right\}. \end{aligned} \quad (13.8)$$

Moreover the following inequality holds:

$$\frac{\partial}{\partial \nu} |r^{(N+2-\frac{2}{p})} \nabla^N U|_{p,S} \leq |r^{(N+2-\frac{2}{p})} L|_{p,S} \quad (13.9)$$

where

$$|L| = \left[N |\hat{\chi} \cdot (\nabla^N U)| + 2 |\hat{\chi} \cdot (\nabla^N \hat{\chi})| + \left| \left\{ (good)_1 \right\} \right| \right] + \left[\frac{(N+2)}{2} |\Omega \text{tr} \chi - \overline{\Omega \text{tr} \chi}| |\nabla^N U| \right]. \quad (13.10)$$

$$\begin{aligned} |r^{(N+2)-\frac{2}{p}} \nabla^N U|_{p,S}(\lambda, \nu) & \leq |r^{(N+2)-\frac{2}{p}} \nabla^N U|_{p,S}(\lambda, \nu_0) \\ & + N \int_{\nu_0}^{\nu} d\nu' |\hat{\chi} \cdot r^{(N+2)-\frac{2}{p}} (\nabla^N U)|_{p,S}(\lambda, \nu') + \frac{(N+2)}{2} \int_{\nu_0}^{\nu} d\nu' |(\Omega \text{tr} \chi - \overline{\Omega \text{tr} \chi}) r^{(N+2)-\frac{2}{p}} (\nabla^N U)|_{p,S}(\lambda, \nu') \\ & + 2 \int_{\nu_0}^{\nu} d\nu' |r^{(N+2)-\frac{2}{p}} \hat{\chi} \cdot (\nabla^N \hat{\chi})|_{p,S}(\lambda, \nu') + \int_{\nu_0}^{\nu} d\nu' |r^{(N+2)-\frac{2}{p}} \left\{ (good)_1 \right\}|_{p,S}(\lambda, \nu'). \end{aligned} \quad (13.11)$$

⁹⁷The analogous estimate for $J = 0$ which we also need is an estimate for $\Omega^{-1} \left(\text{tr} \chi - \frac{\overline{\Omega \text{tr} \chi}}{\Omega} \right)$.

⁹⁸To have the \mathcal{Q}_0 norms bounded.

As already said in the footnote in the proof of Lemma 9.1 it is the term

$$\frac{U}{2} \sum_{k=1}^N \binom{N}{k} (\nabla^k \Omega^2) (\nabla^{N-k} U)$$

which in the transport equation in the interior produces a log divergence forcing us to use Ψ . Here $\nabla^k \Omega^2$ decaying as $r^{-(k+1)-\epsilon}$ avoids the problem and allows to prove in a direct way the estimate⁹⁹

$$|r^{N+2-\frac{2}{p}} \nabla^N U|_{p,S}(\lambda_0, \nu) \leq C_0^{(0)} e^{(N-2)(\delta_0 + \Gamma_0(\lambda))} \frac{N!}{N^\alpha} \frac{1}{\rho_{0,0,1}^N} . \quad (13.12)$$

More important is the fact that in the recursive proof of $\nabla^N \text{tr} \chi$ the Riemann components appear only at order $\nabla^{N-2} \Psi$ which, at its turn, implies that when $\Psi = \beta$, $\nabla^{N-2} \beta$ depends only on $\nabla^{N-1} \text{tr} \chi$ and on $\nabla^{N-2} \zeta$ which we know by inductive assumptions.

13.1.1 The complete control of $\nabla^J \text{tr} \chi$ on C_0

The previous discussion about how we control on C_0 $\nabla^J \text{tr} \chi$ suffers a bit of oversimplification, in fact the situation is slightly more complicated as discussed in [Ca-Ni1], let us recall its main aspect. The transport equation for $U = \Omega^{-1} \text{tr} \chi$,

$$\frac{\partial}{\partial \nu} U + \frac{1}{2} (\Omega \text{tr} \chi) U + \hat{\chi}^{ac} \hat{\chi}^{bd} \gamma_{ab} \gamma_{cd} = 0$$

depends on the S^2 metric γ on C_0 which, at its turn, has to satisfy the transport equation

$$\frac{\partial}{\partial \nu} \gamma_{ab} = (\Omega \text{tr} \chi) \gamma_{ab} + 2\Omega \hat{\chi}_{ab}.$$

Therefore we have to deal with the coupled system of equations,

$$\begin{aligned} \frac{\partial}{\partial \nu} \gamma_{ab} - (\Omega \text{tr} \chi) \gamma_{ab} - 2\Omega \hat{\chi}_{ab} &= 0 \\ \frac{\partial}{\partial \nu} U + \frac{1}{2} (\Omega \text{tr} \chi) U + \hat{\chi}^{ac} \hat{\chi}^{bd} \gamma_{ab} \gamma_{cd} &= 0 . \end{aligned} \quad (13.13)$$

The strategy mimics the one in [Ca-Ni1], we assign $\hat{\chi}$ on C_0 with the due regularity then we solve on C_0 the system 13.13 obtaining γ_{ab} and U (provided we assign them on S_0). Once we control γ the previous discussion shows how to control $\nabla^J U$ for any J on C_0 and, using again the second equation of 13.13, we control all the mixed derivatives of U with respect to ∇ and to \mathcal{D}_ν .

Therefore to complete the initial data construction, also the quantities at the “metric level” have to be assigned. To control the analyticity of the metric

⁹⁹Nevertheless the initial data estimate for $\nabla^J \text{tr} \chi$ cannot have a decay better than $O(r^{-(3+J)})$ and no extra $r^{-\epsilon}$ factor is possible, due to the fact that we integrate starting from S_0 .

components, for instance of γ_{ab} ; the procedure is a little different as, being $\nabla\gamma = 0$, we have to use non covariant derivatives from the beginning. To control the non covariant partial derivatives of γ_{ab} on C_0 we derive the first evolution equation of 13.13,

$$\frac{\partial}{\partial\nu}\partial^J\gamma_{ab} - \sum_{k=0}^J \binom{J}{k} (\partial^k\Omega\text{tr}\chi)\partial^{J-k}\gamma_{ab} - 2\sum_{k=0}^J \binom{J}{k} (\partial^k\Omega)\partial^{J-k}\hat{\chi}_{ab} = 0, \quad (13.14)$$

and proceed in a standard way, recalling that we are able to control of the non covariant partial derivatives for the connection coefficients once we control the covariant ones.¹⁰⁰

Next step is to control on C_0 the norm of ζ and its tangential derivatives; we proceed using the fact that the control of $\nabla^N U$ requires only the inductive assumptions for $\nabla^J\zeta$ with $J \leq N-2$. The evolution equation of ζ along the outgoing cones is¹⁰¹

$$\mathfrak{D}_\nu\zeta + \frac{3}{2}\Omega\text{tr}\chi\zeta + \Omega(\zeta\hat{\chi} - \mathfrak{d}\text{iv}\hat{\chi} + \frac{1}{2}\nabla\text{tr}\chi) + \mathfrak{D}_\nu\nabla\log\Omega = 0. \quad (13.15)$$

From it we easily derive the transport equation along C_0 for $\nabla^N\zeta$,

$$\mathfrak{D}_\nu\nabla^N\zeta - [\mathfrak{D}_\nu, \nabla^N]\zeta = \nabla^N\mathfrak{D}_\nu\zeta = -\nabla^N\left(\frac{3}{2}\Omega\text{tr}\chi\zeta + \Omega(\zeta\hat{\chi} - \mathfrak{d}\text{iv}\hat{\chi} + \frac{1}{2}\nabla\text{tr}\chi) + \mathfrak{D}_\nu\nabla\log\Omega\right), \quad (13.16)$$

therefore

$$\begin{aligned} \mathfrak{D}_\nu\nabla^N\zeta &= -\nabla^N\left(\frac{3}{2}\Omega\text{tr}\chi\zeta + \Omega(\zeta\hat{\chi} - \mathfrak{d}\text{iv}\hat{\chi} + \frac{1}{2}\nabla\text{tr}\chi) + \mathfrak{D}_\nu\nabla\log\Omega\right) + [\mathfrak{D}_\nu, \nabla^N]\zeta \\ &= -3\frac{\Omega\text{tr}\chi}{2}\nabla^N\zeta - \frac{3}{2}\sum_{k=1}^N \binom{N}{k} (\nabla^k\Omega\text{tr}\chi)\nabla^{N-k}\zeta - \nabla^N\left(\Omega(\zeta\hat{\chi} - \mathfrak{d}\text{iv}\hat{\chi} + \frac{1}{2}\nabla\text{tr}\chi + \mathfrak{D}_4\nabla\log\Omega)\right) + [\mathfrak{D}_\nu, \nabla^N]\zeta. \end{aligned} \quad (13.17)$$

As, see Lemma 9.1,

$$\begin{aligned} [\mathfrak{D}_\nu, \nabla^N]\zeta &= \sum_{k=0}^{N-1} \binom{N}{k} (\nabla^{N-1-k}\Omega\chi) \cdot \nabla^{k+1}\zeta - \sum_{k=0}^{N-1} \binom{N+1}{k} (\nabla^{N-1-k}C) \cdot \nabla^k\zeta \\ &= N\frac{\Omega\text{tr}\chi}{2}\nabla^N\zeta + \frac{1}{2}\sum_{k=0}^{N-2} \binom{N}{k} (\nabla^{N-1-k}\Omega\text{tr}\chi) \cdot \nabla^{k+1}\zeta + \sum_{k=0}^{N-1} \binom{N}{k} (\nabla^{N-1-k}\Omega\hat{\chi}) \cdot \nabla^{k+1}\zeta \\ &\quad - \sum_{k=0}^{N-1} \binom{N+1}{k} (\nabla^{N-1-k}C) \cdot \nabla^k\zeta, \end{aligned} \quad (13.18)$$

the final expression for the transport equation is

$$\mathfrak{D}_\nu(\nabla^N\zeta) + \frac{(N+3)}{2}\Omega\text{tr}\chi(\nabla^N\zeta) + \Omega\hat{\chi}(\nabla^N\zeta) = -\nabla^N\left(\Omega(-\mathfrak{d}\text{iv}\hat{\chi} + \frac{1}{2}\nabla\text{tr}\chi + \mathfrak{D}_4\nabla\log\Omega)\right) + \{[l.o.t]\} \quad (13.19)$$

¹⁰⁰Of course in the inductive procedure we do not need to consider the equations for $\nabla^J U$ and $\partial^J\gamma$ as coupled.

¹⁰¹this equation cannot be used for the internal estimates.

where

$$\left\{ [l.o.t] \right\} = \frac{1}{2} \sum_{k=0}^{N-2} \binom{N}{k} (\nabla^{N-1-k} \Omega \text{tr} \chi) \cdot \nabla^{k+1} \zeta + \left\{ -\frac{3}{2} \sum_{k=1}^N \binom{N}{k} (\nabla^k \Omega \text{tr} \chi) \nabla^{N-k} \zeta \right. \\ \left. - \sum_{k=1}^N \binom{N}{k} (\nabla^k \Omega \hat{\chi}) \nabla^{N-k} \zeta + \sum_{k=0}^{N-1} \binom{N}{k} (\nabla^{N-1-k} \Omega \hat{\chi}) \cdot \nabla^{k+1} \zeta - \sum_{k=0}^{N-1} \binom{N+1}{k} (\nabla^{N-1-k} C) \cdot \nabla^k \zeta \right\}.$$

From this equation we obtain the associated equation for the $|\cdot|_{p,S}$ norm and integrating it from below¹⁰² we obtain, see also [Ca-Nil], an inequality of the following kind, with $C_{(J)} = O(1)$ for $J > J_0$, and $O(\varepsilon)$ otherwise,

$$|r^{2+N-\frac{2}{p}} \nabla^N \zeta|_{p,S}(\lambda_0, \nu) \leq C_{(J)} \left(\frac{(N+1)!}{(N+1)^\alpha} \frac{e^{(N-1)(\delta_0 + \Gamma_0(\lambda))}}{\rho_{0,0,1}^{N+1}} \right) \quad (13.20)$$

And, increasing the constant $C_{(J)}$ and exploiting the factor $\frac{1}{(N+1)^\alpha}$ we obtain:

$$|r^{2+N-\frac{2}{p}} \nabla^N \zeta|_{p,S}(\lambda_0, \nu) \leq C_{(J)} \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_{0,0,1}^N} \right) \quad (13.21)$$

assuming as initial data estimates, for $J < N$,

$$|r^{2+J-\frac{2}{p}} \nabla^J \zeta|_{p,S}(\lambda_0, \nu) \leq C_{(J)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_{0,0}^J} \right). \quad (13.22)$$

Remark:

Observe that ζ on C_0 cannot gain an extra $r^{-\varepsilon}$ decay due to the term $\nabla^{N+1} \text{tr} \chi$ present in its transport equation. Nevertheless from the required behavior of σ on $C_0 \cup \underline{C}_0$, see later, we require that on S_0 ζ satisfies, for any J ,

$$|r^{\frac{5}{2}+J+\varepsilon-\frac{2}{p}} \nabla^{J-1} \text{curl} \zeta|_{p,S}(\lambda_0, \nu_0) \leq c \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0 + \Gamma_0)}}{\rho_{0,0}^J} \right). \quad (13.23)$$

To control $\nabla^N \underline{\chi}$ on C_0 we use the transport equation, along C_0 , for $\text{tr} \underline{\chi}$ and for $\hat{\underline{\chi}}$; in this case the loss of derivatives implied by these equations is not harmful as we have already the control of all the ∇ derivatives of ζ (and the ∂ derivatives of γ) which requires only the previous control of χ .

The control of $\underline{\omega}$ and its angular derivatives is, viceversa, a bit more delicate and require that $\underline{\omega}$ satisfies a condition on S_0 as discussed in [Ca-Nil].

More precisely if we try to obtain the initial data norm estimates on C_0 for $\underline{\omega}$ simply requiring that the transport equation,

$$\mathbf{D}_4 \underline{\omega} - 2\omega \underline{\omega} - \zeta \cdot \nabla \log \Omega - \frac{3}{2} |\zeta|^2 + \frac{1}{2} |\nabla \log \Omega|^2 + \frac{1}{2} (\mathbf{K} + \frac{1}{4} \text{tr} \chi \text{tr} \underline{\chi} - \frac{1}{2} \hat{\chi} \cdot \hat{\underline{\chi}}) = 0$$

¹⁰²The initial conditions for ζ and the remaining connection coefficients on S_0 are fully discussed in Section 18.

be satisfied we obtain the estimate than the one we assumed in 3.31,

$$||\lambda_0|^2 r^{J-\frac{2}{p}} \nabla^J \underline{\omega}|_{p,S}(\lambda_0, \nu) \leq C_{(J)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_0^J} \right) \quad (13.24)$$

with $\rho_0 < \rho_{0,0}$ and which a decay apparently in disagreement with the internal estimates in 3.29. This apparent contradiction has the following solution: in the original assignment of the initial data and in the Cauchy-Kowalevski solution in a finite region the double null foliation is not specified and the data are given on $C_0 \cup \underline{C}_0$ in the way we have presented now; in particular $\underline{\omega}$ satisfies the weaker estimate on C_0 , see subsection 9.6.1. As discussed in subsection 9.5, to extend the region and prove the global existence we integrate the transport equations along the outgoing cones from above and in particular we obtain $\underline{\omega}$ on the last slice through the choice of a specific Ω on the last slice and the connection coefficients previously estimated independently from $\underline{\omega}$. In this way the decay estimates for $\underline{\omega}$ in the internal region and, of course also on C_0 are stronger and in particular on C_0 we obtain

$$||\lambda_0| r^{1+J-\frac{2}{p}} \nabla^J \underline{\omega}|_{p,S}(\lambda_0, \nu) \leq C_{(J)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_0^J} \right) .$$

On the other side, as we have a transport equation for $\underline{\omega}$ along the outgoing cones and therefore, also on C_0 , this implies that going, along C_0 from $S(\lambda_0, \nu_*)$ to $S(\lambda_0, \nu_0) \equiv S_0$ the value of $\underline{\omega}$ on S_0 is assigned, see also [Ca-Ni1].

Collecting all these results and the ones for the missing connection coefficients, whose proof we do not report here, being simple extensions of the analogous results in [Ca-Ni1], Lemma 2.3, we obtain,¹⁰³

$$\begin{aligned} |r^{1+J+\sigma(J)-\frac{2}{p}} \nabla^J \text{tr} \underline{\chi}|_{p,S}(\lambda_0, \nu) &\leq C_{(J)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_0^J} \right) \\ ||\lambda_0| r^{1+J-\frac{2}{p}} \nabla^J \underline{\chi}|_{p,S}(\lambda_0, \nu) &\leq C_{(J)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_0^J} \right) \quad (13.25) \\ ||\lambda_0| r^{1+J-\frac{2}{p}} \nabla^J \underline{\omega}|_{p,S}(\lambda_0, \nu) &\leq C_{(J)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_0^J} \right) , \\ |r^{2+J-\frac{2}{p}} \nabla^J \zeta|_{p,S}(\lambda_0, \nu) &\leq C_{(J)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_0^J} \right) , \end{aligned}$$

with $\rho_0 < \rho_{0,0,1} < \rho_{0,0}$. Observe that we can also conclude that on C_0 all the remaining connection coefficients satisfy the following estimates, for any J ,¹⁰⁴

$$|r^{1+J+\sigma(J)-\frac{2}{p}} \nabla^J \text{tr} \chi|_{p,S}(\lambda_0, \nu) \leq C_{(J)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_0^J} \right)$$

¹⁰³If, as discussed above, we assume an arbitrary foliation on \underline{C}_0 , not the specific one induced by the canonical foliation, the estimate on C_0 for $\underline{\omega}$ would be the weaker one, 13.24.

¹⁰⁴Recalling, anyway, that the first connection coefficients can have a better estimate, with $\rho_{0,0,1} > \rho_0$.

$$\begin{aligned}
|r^{\frac{5}{2}+J+\epsilon-\frac{2}{p}} \nabla^J \hat{\chi}|_{p,S}(\lambda_0, \nu) &\leq C_{(J)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^J} \right) \\
|r^{2+J+\epsilon-\frac{2}{p}} \nabla^J \omega|_{p,S}(\lambda_0, \nu_0) &\leq C_2^{(0)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^J} \right)
\end{aligned} \tag{13.26}$$

proving the first part of Theorem 1.1.

Remark:

Observe that $\hat{\chi}$ on C_0 cannot gain an extra $r^{-\epsilon}$ decay due to the factor $\frac{1}{2} \text{tr} \chi \hat{\chi}$ present in its transport equation. Nevertheless as required from the estimates of $\hat{\chi}$ on \underline{C}_0 , see later, we ask that on S_0 $\hat{\chi}$ satisfies, for any J ,

$$|r_0^{\frac{5}{2}+J+\epsilon-\frac{2}{p}} \nabla^J \hat{\chi}|_{p,S}(\lambda_0, \nu_0) \leq c_{(J)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0+\Gamma_{0,0})}}{\rho_{0,0}^J} \right). \tag{13.27}$$

13.2 The initial data on the incoming cone

13.2.1 The connection coefficient ζ on the initial incoming cone

Observe first of all that on \underline{C}_0 $\hat{\chi}$, X and Ω , are assigned in a free way therefore also $\underline{\omega}$ is assigned freely on \underline{C}_0 , the requirement on the norms for these quantities are similar to those for $\hat{\chi}$ and Ω on C_0 namely,

$$\begin{aligned}
||\lambda|^{1+\epsilon} r^{1+J-\frac{2}{p}} \nabla^J \log \Omega|_{p,S} &\leq \underline{C}_3^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,0,1}^J} \\
||\lambda|^{\frac{3}{2}+\epsilon} r^{1+J-\frac{2}{p}} \nabla^J \hat{\chi}|_{p,S} &\leq \underline{C}_1^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,0,1}^J} \\
||\lambda|^{\frac{3}{2}+\epsilon} r^{1+J-\frac{2}{p}} \nabla^J X|_{p,S} &\leq \underline{C}_1^{(0)} e^{(J-2)(\delta_0+\Gamma_0(\lambda))} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,0,1}^J}.
\end{aligned} \tag{13.28}$$

Proceeding as for $\text{tr} \chi$ we control $\nabla^J \text{tr} \chi$ on \underline{C}_0 . The next step is to control ζ whose transport equation is on \underline{C}_0

$$\mathcal{D}_3 \zeta + \frac{3}{2} \text{tr} \chi \zeta + \zeta \cdot \hat{\chi} + \text{div} \chi - \nabla \text{tr} \chi - \mathcal{D}_3 \nabla \log \Omega = 0. \tag{13.29}$$

The factor $3/2$ is a problem as it implies that we have a transport equation for $|r^{N+3-\frac{2}{p}} \nabla^N \zeta|_{p,S}$, while we expect that the possible bound is for $|r^{N+2-\frac{2}{p}} \nabla^N \zeta|_{p,S}$ and to lower the weight factor is problematic as, differently from the C_0 case, moving along the incoming cone starting from S_0 the radius of S decreases. We proceed in a different way writing the transport equation for ζ along \underline{C}_0 ,

$$\mathcal{D}_3 \zeta + \text{tr} \chi \zeta + 2\zeta \cdot \hat{\chi} - (\nabla \text{tr} \chi - \text{div} \chi - \zeta \text{tr} \chi + \zeta \cdot \chi) - \mathcal{D}_3 \nabla \log \Omega = 0. \tag{13.30}$$

as

$$\mathcal{D}_3 \zeta + \text{tr} \chi \zeta + 2\zeta \cdot \hat{\chi} - \mathcal{D}_3 \nabla \log \Omega + \underline{\beta} = 0 \tag{13.31}$$

using the relation,

$$\underline{\beta} = -(\nabla \text{tr} \underline{\chi} - \text{div} \underline{\chi} - \zeta \text{tr} \underline{\chi} + \zeta \cdot \underline{\chi}) . \quad (13.32)$$

The crucial remark is that on \underline{C}_0 all the factors composing $\underline{\beta}$ are already under control with the exception of ζ , therefore we solve simultaneously the two transport equations for ζ and $\underline{\beta}$,

$$\begin{aligned} \mathfrak{D}_3 \zeta + \text{tr} \underline{\chi} \zeta + 2\zeta \cdot \underline{\hat{\chi}} + \underline{\beta} - (\mathfrak{D}_3 \nabla \log \Omega) &= 0 \\ \mathfrak{D}_3 \underline{\beta} + 2\text{tr} \underline{\chi} \underline{\beta} + 2\underline{\omega} \underline{\beta} - \zeta \cdot \underline{\alpha} - (\text{div} \underline{\alpha} + \nabla \log \Omega \cdot \underline{\alpha}) &= 0 , \end{aligned} \quad (13.33)$$

where it is crucial to observe that

$$\underline{\alpha} = -(\mathfrak{D}_3 \underline{\hat{\chi}} + \text{tr} \underline{\chi} + 2\underline{\omega} \underline{\hat{\chi}}) \quad (13.34)$$

can be considered as a given data as, on \underline{C}_0 , $\underline{\chi}$ and $\underline{\omega}$ are already assigned. These two equations can be rewritten in a slightly different way as

$$\begin{aligned} \mathfrak{D}_\lambda \zeta + \Omega \text{tr} \underline{\chi} \zeta + 2\Omega \underline{\hat{\chi}} \cdot \zeta + \Omega \underline{\beta} - (\mathfrak{D}_\lambda \nabla \log \Omega) &= 0 \\ \mathfrak{D}_\lambda \underline{\beta} + 2\Omega \text{tr} \underline{\chi} \underline{\beta} + 2\Omega \underline{\omega} \underline{\beta} - \Omega \underline{\alpha} \cdot \zeta - [\Omega(\text{div} \underline{\alpha} + \nabla \log \Omega \cdot \underline{\alpha})] &= 0 . \end{aligned} \quad (13.35)$$

We expect that the solution to this system produces the correct estimates for the ζ and $\underline{\beta}$ norms. In fact we prove the following lemma

Lemma 13.1 *Assuming that on \underline{C}_0 the following estimates hold with $\tilde{\delta} > \delta \geq 0$,*

$$\mathfrak{D}_\lambda \nabla \Omega = O\left(r^{-(2+\epsilon)} |\lambda|^{-(1+\tilde{\delta})}\right) , \quad \underline{\alpha} = O\left(r^{-(1+\epsilon)} |\lambda|^{-(\frac{5}{2}+\tilde{\delta})}\right) . \quad (13.36)$$

and that on S_0 we have

$$|r^{2-\frac{2}{p}} \zeta|_{p,S}(\lambda_0, \nu_0) \leq \varepsilon , \quad ||\lambda|^{\frac{3}{2}} r^{2+\epsilon-\frac{2}{p}} \underline{\beta}|_{p,S}(\lambda_0, \nu_0) \leq \varepsilon . \quad (13.37)$$

then the following estimates hold on \underline{C}_0

$$|r^{2-\frac{2}{p}} \zeta|_{p,S}(\lambda, \nu_0) \leq \varepsilon , \quad ||\lambda|^{\frac{3}{2}} r^{2+\epsilon-\frac{2}{p}} \underline{\beta}|_{p,S}(\lambda, \nu_0) \leq \varepsilon . \quad (13.38)$$

Proof: See Appendix 18 .

As before we do not report the proofs for the decays of the other connection coefficients, being them only a repetition of the ones given in [Ca-Nil]

13.2.2 The control of the required decay for the Riemann components on C_0

To complete the discussion about the initial data on C_0 we have to show that the proved results allow to fulfill the asymptotic behaviour 7.1. Let us start to consider α whose expression is¹⁰⁵

$$\alpha = -[\mathfrak{D}_4 \underline{\hat{\chi}} + \text{tr} \chi \underline{\hat{\chi}} - (\mathbf{D}_4 \log \Omega) \underline{\hat{\chi}}] .$$

¹⁰⁵ α (and $\underline{\alpha}$) plays a different role between the null Riemann components. In fact here its behavior can be assigned simply assigning Ω and $\underline{\hat{\chi}}$ on C_0 . The same does not happen for β as, in its case, $\text{tr} \chi$ and ζ cannot be assigned freely as they have to satisfy some constraint equations, see the discussion in (I).

The expected asymptotic behavior for α is satisfied provided that the estimates 13.1 for $\hat{\chi}$ and Ω on C_0 are satisfied. To prov the expected asymptotic behavior for β requires more work and an extra condition; in fact its explicit expression,

$$\beta = \nabla \text{tr} \chi - \text{div} \chi - \zeta \cdot \chi + \zeta \text{tr} \chi ,$$

seems not to provide the right decay, but if we integrate its evolution equation along the outgoing cone, C_0 ,

$$\mathfrak{D}_\nu \beta + 2\Omega \text{tr} \chi \beta = 2\Omega \omega \beta - \Omega [\text{div} \alpha - (\zeta + \nabla \log \Omega) \alpha] ,$$

we obtain

$$|r^{4-\frac{2}{p}} \beta|_{p,S}(\lambda_0, \nu) \leq c \left(|r^{4-\frac{2}{p}} \beta|_{p,S}(\lambda_0, \nu_0) + \int_{\nu_0}^{\nu} d\nu' \left(|r^{4-\frac{2}{p}} \Omega \zeta \cdot \alpha|_{p,S} + |r^{4-\frac{2}{p}} L_2|_{p,S} \right)(\lambda_0, \nu') \right)$$

where

$$L_2 = \Omega (\text{div} \alpha - \nabla \log \Omega \cdot \alpha) .$$

Dividing both sides by $r^{(\frac{1}{2}-\epsilon)}(\lambda_0, \nu)$ and recalling that, for $\nu' \leq \nu$,

$$r(\lambda_0, \nu)^{-1} < r(\lambda_0, \nu')^{-1} < r(\lambda_0, \nu_0)^{-1} ,$$

we obtain

$$|r^{\frac{7}{2}+\epsilon-\frac{2}{p}} \beta|_{p,S}(\lambda_0, \nu) \leq c \left(|r^{\frac{7}{2}+\epsilon-\frac{2}{p}} \beta|_{p,S}(\lambda_0, \nu_0) + \frac{1}{r^{(\frac{1}{2}-\epsilon)}(\lambda_0, \nu)} \int_{\nu_0}^{\nu} d\nu' \left(|r^{4-\frac{2}{p}} \Omega \zeta \cdot \alpha|_{p,S} + |r^{4-\frac{2}{p}} L_2|_{p,S} \right) \right)$$

and the integral part is bounded due to the previous estimates. The same procedure can be applied to estimate all the angular derivatives of β .¹⁰⁶ It remains, nevertheless, to prove that on S_0 we have

$$|r^{\frac{7}{2}+\epsilon-\frac{2}{p}} \beta|_{p,S}(\lambda_0, \nu_0) \leq c\varepsilon .$$

Looking again at the expression for β in terms of the connection coefficients, 13.39, on S_0 the following condition must hold, where $r_0 = r(\lambda_0, \nu_0) = |\lambda_0| = \nu_0$,

$$\nabla \text{tr} \chi - \zeta \text{tr} \chi = O(\varepsilon r_0^{-(\frac{7}{2}+\epsilon)}) . \quad (13.39)$$

Next step is devoted to the the decay of the null Riemann components ρ and σ ; from their evolution equations and their explicit expressions, see also [Ca-Nil], it follows that we obtain the required decay for $(\rho - \bar{\rho})$ provided that on S_0

$$\mathbf{K} - \bar{\mathbf{K}} + \frac{1}{4}(\text{tr} \chi \text{tr} \underline{\chi} - \overline{\text{tr} \chi \text{tr} \underline{\chi}}) = O(\varepsilon r_0^{-\frac{7}{2}}) . \quad (13.40)$$

¹⁰⁶In fact to all the tangential derivatives on C_0 , therefore also for the \mathfrak{D}_ν derivatives, this last problem has already been treated in the general case and we do not have to repeat here.

and repeating the same procedure for σ we obtain its correct decay provided that on S_0 the following condition holds,¹⁰⁷

$$\text{cyl}\zeta = O(\varepsilon r_0^{-\frac{7}{2}}) . \quad (13.41)$$

Finally the same procedure used for estimating β holds for $\underline{\beta}$,¹⁰⁸ obtaining the appropriate decay requiring that, on S_0 , the following condition is satisfied

$$\nabla \text{tr}\underline{\chi} + \zeta \text{tr}\underline{\chi} = O(\varepsilon r_0^{-(\frac{7}{2}+\epsilon)}) . \quad (13.42)$$

Summarizing we need that on S_0 the initial data satisfy the following conditions,

$$\begin{aligned} \nabla \text{tr}\chi - \zeta \text{tr}\chi &= O(\varepsilon r_0^{-(\frac{7}{2}+\epsilon)}) \\ \mathbf{K} - \overline{\mathbf{K}} + \frac{1}{4}(\text{tr}\chi \text{tr}\underline{\chi} - \overline{\text{tr}\chi \text{tr}\underline{\chi}}) &= O(\varepsilon r_0^{-\frac{7}{2}}) \\ \text{cyl}\zeta &= O(\varepsilon r_0^{-\frac{7}{2}}) . \end{aligned} \quad (13.43)$$

13.3 The mixed derivatives for the initial data. proof of theorem 3.4:

13.3.1 Some useful computations:

We have to compute $[\mathcal{D}_\nu, \mathcal{D}_\lambda]$, let $U = U_{\sigma_1 \nu_2 \dots \sigma_k} d\omega^{\sigma_1} \otimes d\omega^{\sigma_2} \otimes \dots \otimes d\omega^{\sigma_k}$ a $(0, k)$ S -tangent tensor, we have

$$\mathcal{D}_\nu \mathcal{D}_\lambda U = \Omega \mathcal{D}_4 \Pi_{\nu_1 \dots \nu_k}^{\sigma_1 \dots \sigma_k} \Omega \mathcal{D}_3 U_{\sigma_1 \dots \sigma_k} = \Omega \Pi_{\tau_1 \dots \tau_k}^{\nu_1 \dots \nu_k} \mathcal{D}_4 \Pi_{\nu_1 \dots \nu_k}^{\sigma_1 \dots \sigma_k} \Omega \mathcal{D}_3 U_{\sigma_1 \dots \sigma_k} \quad (13.44)$$

$$\mathcal{D}_\lambda \mathcal{D}_\nu U = \Omega \mathcal{D}_3 \Pi_{\nu_1 \dots \nu_k}^{\sigma_1 \dots \sigma_k} \Omega \mathcal{D}_4 U_{\sigma_1 \dots \sigma_k} = \Omega \Pi_{\tau_1 \dots \tau_k}^{\nu_1 \dots \nu_k} \mathcal{D}_3 \Pi_{\nu_1 \dots \nu_k}^{\sigma_1 \dots \sigma_k} \Omega \mathcal{D}_4 U_{\sigma_1 \dots \sigma_k} . \quad (13.45)$$

We have

$$\begin{aligned} & \Pi_{\tau_1 \dots \tau_k}^{\nu_1 \dots \nu_k} \mathcal{D}_4 \Pi_{\nu_1 \dots \nu_k}^{\sigma_1 \dots \sigma_k} (\Omega \mathcal{D}_3 U)_{\sigma_1 \dots \sigma_j \dots \sigma_k} \\ &= \sum_{j=1}^k \Pi_{\tau_1 \dots \hat{\tau}_j \dots \tau_k}^{\sigma_1 \dots \hat{\sigma}_j \dots \sigma_k} (\Pi_{\tau_j}^{\nu_j} \mathcal{D}_4 \Pi_{\nu_j}^{\sigma_j}) (\Omega \mathcal{D}_3 U)_{\sigma_1 \dots \sigma_j \dots \sigma_k} + \Pi_{\tau_1 \dots \tau_k}^{\sigma_1 \dots \sigma_k} \mathcal{D}_4 (\Omega \mathcal{D}_3 U)_{\sigma_1 \dots \sigma_k} \\ &= - \sum_{j=1}^k \Pi_{\tau_1 \dots \hat{\tau}_j \dots \tau_k}^{\sigma_1 \dots \hat{\sigma}_j \dots \sigma_k} \left(\Pi_{\tau_j}^{\nu_j} (\mathcal{D}_4 \theta^4)_{\nu_j} e_4^{\sigma_j} + \Pi_{\tau_j}^{\nu_j} (\mathcal{D}_4 \theta^3)_{\nu_j} e_3^{\sigma_j} \right) (\Omega \mathcal{D}_3 U)_{\sigma_1 \dots \sigma_j \dots \sigma_k} \end{aligned}$$

¹⁰⁷We already know from the previous requirements that on S_0 , $(\hat{\chi} \wedge \hat{\chi})$ behaves much better, as $r_0^{-(\frac{11}{2}+\epsilon)}$.

¹⁰⁸Analogous condition is not required for $\underline{\alpha}$ on C_0 as $\underline{\alpha}$ is not present in the part of \mathcal{Q}_0 made by an integration along C_0 .

$$\begin{aligned}
& + \Pi_{\tau_1 \dots \tau_k}^{\sigma_1 \dots \sigma_k} \mathbf{D}_4(\Omega \mathbf{D}_3 U)_{\sigma_1 \dots \sigma_k} \\
& = - \sum_{j=1}^k \Pi_{\tau_1 \dots \hat{\tau}_j \dots \tau_k}^{\sigma_1 \dots \hat{\sigma}_j \dots \sigma_k} \left(-\frac{1}{2} \mathbf{g}(\mathbf{D}_4 e_3, e_C) \theta_{\tau_j}^C \right) e_4^{\sigma_j} (\Omega \mathbf{D}_3 U)_{\sigma_1 \dots \sigma_j \dots \sigma_k} + \Pi_{\tau_1 \dots \tau_k}^{\sigma_1 \dots \sigma_k} \mathbf{D}_4(\Omega \mathbf{D}_3 U)_{\sigma_1 \dots \sigma_k} \\
& = -\frac{1}{2} \sum_{j=1}^k \Pi_{\tau_1 \dots \hat{\tau}_j \dots \tau_k}^{\sigma_1 \dots \hat{\sigma}_j \dots \sigma_k} \left(\mathbf{g}(\mathbf{D}_4 e_3, e_C) \theta_{\tau_j}^C \right) (\Omega \mathbf{D}_3 e_4)^{\sigma_j} U_{\sigma_1 \dots \sigma_j \dots \sigma_k} + \Pi_{\tau_1 \dots \tau_k}^{\sigma_1 \dots \sigma_k} \mathbf{D}_4(\Omega \mathbf{D}_3 U)_{\sigma_1 \dots \sigma_k} \\
& = -\frac{1}{2} \sum_{j=1}^k \Pi_{\tau_1 \dots \hat{\tau}_j \dots \tau_k}^{\sigma_1 \dots \hat{\sigma}_j \dots \sigma_k} \underline{\eta}_{\tau_j} (\Omega \mathbf{D}_3 e_4)^{\sigma_j} U_{\sigma_1 \dots \sigma_j \dots \sigma_k} + \Pi_{\tau_1 \dots \tau_k}^{\sigma_1 \dots \sigma_k} \mathbf{D}_4(\Omega \mathbf{D}_3 U)_{\sigma_1 \dots \sigma_k}
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathfrak{P}_\nu \mathfrak{P}_\lambda U & = \Omega \mathfrak{P}_4 \Pi_{\tau_1 \dots \tau_k}^{\sigma_1 \dots \sigma_k} \Omega \mathbf{D}_3 U_{\sigma_1 \dots \sigma_k} = \Omega \Pi_{\tau_1 \dots \tau_k}^{\nu_1 \dots \nu_k} \mathbf{D}_4 \Pi_{\nu_1 \dots \nu_k}^{\sigma_1 \dots \sigma_k} \Omega \mathbf{D}_3 U_{\sigma_1 \dots \sigma_k} \quad (13.46) \\
& = -\frac{\Omega}{2} \sum_{j=1}^k \Pi_{\tau_1 \dots \hat{\tau}_j \dots \tau_k}^{\sigma_1 \dots \hat{\sigma}_j \dots \sigma_k} \underline{\eta}_{\tau_j} (\Omega \mathbf{D}_3 e_4)^{\sigma_j} U_{\sigma_1 \dots \sigma_j \dots \sigma_k} + \Pi_{\tau_1 \dots \tau_k}^{\sigma_1 \dots \sigma_k} \Omega \mathbf{D}_4(\Omega \mathbf{D}_3 U)_{\sigma_1 \dots \sigma_k} \\
& = -\Omega^2 \sum_{j=1}^k \Pi_{\tau_1 \dots \hat{\tau}_j \dots \tau_k}^{\sigma_1 \dots \hat{\sigma}_j \dots \sigma_k} \underline{\eta}_{\tau_j} (\eta \cdot U)_{\sigma_1 \dots \sigma_j \dots \sigma_k} + \Pi_{\tau_1 \dots \tau_k}^{\sigma_1 \dots \sigma_k} \Omega \mathbf{D}_4(\Omega \mathbf{D}_3 U)_{\sigma_1 \dots \sigma_k}
\end{aligned}$$

Analogously

$$\begin{aligned}
\mathfrak{P}_\lambda \mathfrak{P}_\nu U & = \Omega \mathfrak{P}_3 \Pi_{\tau_1 \dots \tau_k}^{\sigma_1 \dots \sigma_k} \Omega \mathbf{D}_4 U_{\sigma_1 \dots \sigma_k} = \Omega \Pi_{\tau_1 \dots \tau_k}^{\nu_1 \dots \nu_k} \mathbf{D}_4 \Pi_{\nu_1 \dots \nu_k}^{\sigma_1 \dots \sigma_k} \Omega \mathbf{D}_3 U_{\sigma_1 \dots \sigma_k} \quad (13.47) \\
& = -\frac{\Omega}{2} \sum_{j=1}^k \Pi_{\tau_1 \dots \hat{\tau}_j \dots \tau_k}^{\sigma_1 \dots \hat{\sigma}_j \dots \sigma_k} \eta_{\tau_j} (\Omega \mathbf{D}_4 e_3)^{\sigma_j} U_{\sigma_1 \dots \sigma_j \dots \sigma_k} + \Pi_{\tau_1 \dots \tau_k}^{\sigma_1 \dots \sigma_k} \Omega \mathbf{D}_3(\Omega \mathbf{D}_4 U)_{\sigma_1 \dots \sigma_k} \\
& = -\Omega^2 \sum_{j=1}^k \Pi_{\tau_1 \dots \hat{\tau}_j \dots \tau_k}^{\sigma_1 \dots \hat{\sigma}_j \dots \sigma_k} \eta_{\tau_j} (\underline{\eta} \cdot U)_{\sigma_1 \dots \sigma_j \dots \sigma_k} + \Pi_{\tau_1 \dots \tau_k}^{\sigma_1 \dots \sigma_k} \Omega \mathbf{D}_3(\Omega \mathbf{D}_4 U)_{\sigma_1 \dots \sigma_k} .
\end{aligned}$$

Therefore

$$\begin{aligned}
[\mathfrak{P}_\nu, \mathfrak{P}_\lambda] U_{\tau_1 \dots \tau_k} & = -\Omega^2 \sum_{j=1}^k \Pi_{\tau_1 \dots \hat{\tau}_j \dots \tau_k}^{\sigma_1 \dots \hat{\sigma}_j \dots \sigma_k} \left(\underline{\eta}_{\tau_j} (\eta \cdot U) - \eta_{\tau_j} (\underline{\eta} \cdot U) \right)_{\sigma_1 \dots \hat{\sigma}_j \dots \sigma_k} + \Pi_{\tau_1 \dots \tau_k}^{\sigma_1 \dots \sigma_k} [\mathbf{D}_\nu, \mathbf{D}_\lambda] U_{\sigma_1 \dots \sigma_k} \\
& = -\Omega^2 \sum_{j=1}^k \left(\underline{\eta}_{\tau_j} (\eta \cdot U) - \eta_{\tau_j} (\underline{\eta} \cdot U) \right)_{\tau_1 \dots \hat{\tau}_j \dots \tau_k} \quad (13.48)
\end{aligned}$$

$$\begin{aligned}
& + \Omega^2 \sum_{i=1}^k R^{\tilde{\tau}_i}(e_4, e_3)_{\tau_i} U_{\tau_1 \dots \hat{\tau}_i \dots \tau_k} + \Pi_{\tau_1 \dots \tau_k}^{\sigma_1 \dots \sigma_k} [\Omega e_4, \Omega e_3]^\mu D_\mu U_{\sigma_1 \dots \sigma_k} \\
& = -\Omega^2 \sum_{j=1}^k \left(\underline{\eta}_{\tau_j} (\eta \cdot U) - \eta_{\tau_j} (\underline{\eta} \cdot U) \right)_{\tau_1 \dots \hat{\tau}_j \dots \tau_k} \quad (13.49) \\
& + \Omega^2 \sum_{i=1}^k R^{\tilde{\tau}_i}(e_4, e_3)_{\tau_i} U_{\tau_1 \dots \hat{\tau}_i \dots \tau_k} + 4\Omega^2 (\zeta \cdot \nabla U)_{\tau_1 \dots \tau_k} .
\end{aligned}$$

Therefore

$$\begin{aligned}
& \sum_{k=0}^{J-1} \mathfrak{P}_\nu^{J-1-k} [\mathfrak{P}_\nu, \mathfrak{P}_\lambda] \mathfrak{P}_\nu^k \nabla^P \hat{\omega}_{\tau_1 \dots \tau_P} \\
&= \sum_{k=0}^{J-1} \mathfrak{P}_\nu^{J-1-k} \left[-\Omega^2 \sum_{j=1}^P \left(\underline{\eta}_{\tau_j} (\eta \cdot \mathfrak{P}_\nu^k \nabla^P \hat{\omega})_{\tau_1 \dots \tilde{\tau}_j \dots \tau_P} - \eta_{\tau_j} (\underline{\eta} \cdot \mathfrak{P}_\nu^k \nabla^P \hat{\omega})_{\tau_1 \dots \tilde{\tau}_j \dots \tau_P} \right) \right. \\
&\quad \left. + \Omega^2 \sum_{j=1}^P R^{\tilde{\tau}_j}(e_4, e_3)_{\tau_j} (\mathfrak{P}_\nu^k \nabla^P \hat{\omega})_{\tau_1 \dots \tilde{\tau}_j \dots \tau_P} + 4\Omega^2 (\zeta \cdot \nabla (\mathfrak{P}_\nu^k \nabla^P \hat{\omega})_{\tau_1 \dots \tau_P} \right] ,
\end{aligned} \tag{13.50}$$

which we rewrite as

$$\begin{aligned}
\sum_{k=0}^{J-1} \mathfrak{P}_\nu^{J-1-k} [\mathfrak{P}_\nu, \mathfrak{P}_\lambda] \mathfrak{P}_\nu^k \nabla^P \hat{\omega} &= \sum_{k=0}^{J-1} \mathfrak{P}_\nu^{J-1-k} \left[-\Omega^2 P \left(\underline{\eta} \otimes (\eta \cdot \mathfrak{P}_\nu^k \nabla^P \hat{\omega}) - \eta \otimes (\underline{\eta} \cdot \mathfrak{P}_\nu^k \nabla^P \hat{\omega}) \right) \right. \\
&\quad \left. + \Omega^2 P R(e_4, e_3) \cdot (\mathfrak{P}_\nu^k \nabla^P \hat{\omega}) + 4\Omega^2 \zeta \cdot \nabla (\mathfrak{P}_\nu^k \nabla^P \hat{\omega}) \right] .
\end{aligned} \tag{13.51}$$

Let us write explicitly the term

$$\sum_{h=0}^{P-1} \mathfrak{P}_\nu^J \nabla^{P-1-h} [\nabla, \mathfrak{P}_\lambda] \nabla^h \hat{\omega} ,$$

we recall that, as done in the appendix, see 14.1,

$$\begin{aligned}
[\nabla_\mu, \mathfrak{P}_\lambda] U_{\nu_1 \dots \nu_k} &= -\Omega \left[\sum_{j=1}^k \left(\underline{\chi}_{\mu\nu_j} \eta^{\sigma_j} - \underline{\chi}_\mu^{\sigma_j} \eta_{\nu_j} \right) U_{\nu_1 \dots \sigma_j \dots \nu_k} - \underline{\chi}_\mu^\rho (\nabla_\rho U)_{\nu_1 \dots \nu_k} \right] \\
&\quad - \Omega \theta_\mu^C \sum_{j=1}^k \theta_{\nu_j}^D R^{\sigma_j}(\cdot, e_C, e_3, e_D) U_{\nu_1 \dots \sigma_j \dots \nu_k} \\
&= -\Omega \sum_{j=1}^k \left[(\underline{\chi}_{\mu\nu_j} \eta^{\sigma_j} - \underline{\chi}_\mu^{\sigma_j} \eta_{\nu_j}) + \theta_\mu^C \theta_{\nu_j}^D R^{\sigma_j}(\cdot, e_C, e_3, e_D) \right] U_{\nu_1 \dots \sigma_j \dots \nu_k} + \Omega \underline{\chi}_\mu^\rho (\nabla_\rho U)_{\nu_1 \dots \nu_k} \\
&= -\sum_{j=1}^k \Omega \underline{\mathcal{C}}_{\mu\nu_j}^{\sigma_j} U_{\nu_1 \dots \sigma_j \dots \nu_k} + \Omega \underline{\chi}_\mu^\rho (\nabla_\rho U)_{\nu_1 \dots \nu_k}
\end{aligned} \tag{13.52}$$

where

$$\underline{\mathcal{C}}_{\mu\nu_j}^{\sigma_j} = (\underline{\chi}_{\mu\nu_j} \eta^{\sigma_j} - \underline{\chi}_\mu^{\sigma_j} \eta_{\nu_j}) + \theta_\mu^C \theta_{\nu_j}^D R^{\sigma_j}(\cdot, e_C, e_3, e_D) . \tag{13.53}$$

Therefore

$$[\nabla_\mu, \mathfrak{P}_\lambda] (\nabla^h \hat{\omega})_{\nu_1 \dots \nu_h} = -\sum_{j=1}^h \underline{\mathcal{C}}_{\mu\nu_j}^{\sigma_j} (\nabla^h \hat{\omega})_{\nu_1 \dots \sigma_j \dots \nu_h} + \Omega \underline{\chi}_\mu^\rho (\nabla_\rho \nabla^h \hat{\omega})_{\nu_1 \dots \nu_h} \tag{13.54}$$

and

$$\sum_{h=0}^{P-1} \mathfrak{P}_\nu^J \nabla^{P-1-h} [\nabla, \mathfrak{P}_\lambda] \nabla^h \hat{\omega}_{\nu_1 \dots \nu_h} = \sum_{h=0}^{P-1} \mathfrak{P}_\nu^J \nabla^{P-1-h} \left[- \sum_{j=1}^h \underline{C}_{\mu\nu_j}^{\sigma_j} (\nabla^h \hat{\omega})_{\nu_1 \dots \sigma_j \dots \nu_h} + \Omega \underline{\chi}_\mu^\rho (\nabla_\rho \nabla^h \hat{\omega})_{\nu_1 \dots \nu_h} \right]$$

which we rewrite in a more compact way as

$$\begin{aligned} \sum_{h=0}^{P-1} \mathfrak{P}_\nu^J \nabla^{P-1-h} [\nabla, \mathfrak{P}_\lambda] \nabla^h \hat{\omega} &= \sum_{h=0}^{P-1} \mathfrak{P}_\nu^J \nabla^{P-1-h} \left[-\Omega h \left(\underline{\chi} \otimes (\eta \cdot \nabla^h \hat{\omega}) - \eta \otimes (\underline{\chi} \cdot \nabla^h \hat{\omega}) \right) \right. \\ &\quad \left. -\Omega h R(\cdot, \cdot, e_3, \cdot) \cdot (\nabla^h \hat{\omega}) + \Omega \underline{\chi} \cdot \nabla (\nabla^h \hat{\omega}) \right] . \end{aligned} \quad (13.55)$$

In conclusion we have

$$\begin{aligned} \sum_{k=0}^{J-1} \mathfrak{P}_\nu^{J-1-k} [\mathfrak{P}_\nu, \mathfrak{P}_\lambda] \mathfrak{P}_\nu^k \nabla^P \hat{\omega} &= \sum_{k=0}^{J-1} \mathfrak{P}_\nu^{J-1-k} \left[\Omega^2 P \left(\underline{\eta} \otimes (\eta \cdot \mathfrak{P}_\nu^k \nabla^P \hat{\omega}) - \eta \otimes (\underline{\eta} \cdot \mathfrak{P}_\nu^k \nabla^P \hat{\omega}) \right) \right. \\ &\quad \left. + \Omega^2 P R(e_4, e_3) \cdot (\mathfrak{P}_\nu^k \nabla^P \hat{\omega}) + 4\Omega^2 \zeta \cdot \nabla (\mathfrak{P}_\nu^k \nabla^P \hat{\omega}) \right] \\ \sum_{h=0}^{P-1} \mathfrak{P}_\nu^J \nabla^{P-1-h} [\nabla, \mathfrak{P}_\lambda] \nabla^h \hat{\omega} &= \sum_{h=0}^{P-1} \mathfrak{P}_\nu^J \nabla^{P-1-h} \left[-\Omega h \left(\underline{\chi} \otimes (\eta \cdot \nabla^h \hat{\omega}) - \eta \otimes (\underline{\chi} \cdot \nabla^h \hat{\omega}) \right) \right. \\ &\quad \left. -\Omega h R(\cdot, \cdot, e_3, \cdot) \cdot (\nabla^h \hat{\omega}) + \Omega \underline{\chi} \cdot \nabla (\nabla^h \hat{\omega}) \right] . \end{aligned} \quad (13.56)$$

13.4 proof of Theorem 3.4

Theorem 3.4

Assuming the estimates 3.15 and 3.16 for any N , and the estimates 13.1, 13.28 for $\{\Omega, \chi, \underline{\chi}\}$ on $C_0 \cup \underline{C}_0$, the estimates 3.21 for the $\mathfrak{P}_\nu^J \nabla^P$ for $(J-1, P+1)$ with $J+P \leq N$, then the estimates 3.21 hold for $J+P \leq N$ on $C_0 \cup \underline{C}_0$ with the various F constants $O(1)$.

Proof: First of all, by the Bianchi equations, see [Kl-Ni2] equations 2.2.13, the inductive assumptions and exploiting those of the structure equation which depends on the null Riemann components, see [Kl-Ni2] equations 3.1.46, we can assume the following estimate on the null Riemann components.

$$\begin{aligned} ||\lambda|_{\underline{\phi}(\Psi)}^{\phi(\Psi)} r^{\phi(\Psi)+N-1-\frac{2}{p}} \mathfrak{P}_\nu^{J-1} \nabla^P \Psi|_{L^p(S)} &\leq C^{(1)} e^{(J-3)\delta_0} e^{(J-3)\Gamma_0(\lambda)} \frac{N-1!}{(N-1)^\alpha} \frac{1}{\rho_0^{N-1}} , \\ ||\lambda|_{\underline{\phi}(\Psi)+J-1}^{\phi(\Psi)+J-1} r^{\phi(\Psi)+P-\frac{2}{p}} \mathfrak{P}_\lambda^{J-1} \nabla^P \Psi|_{L^p(S)} &\leq C^{(1)} e^{(J-3)\delta_0} e^{(J-3)\Gamma_0(\lambda)} \frac{N-1!}{(N-1)^\alpha} \frac{1}{\rho_0^{N-1}} \end{aligned} \quad (13.57)$$

We go by induction on the number of \mathfrak{P}_ν derivatives. The base of induction is nothing more that estimates 3.15 and 3.16 of TheoremThinitialdata.

First of all notice that we already have the right estimates for $\mathfrak{D}_\nu^J \nabla^P \hat{\chi}$, $\mathfrak{D}_\nu^J \nabla^P \underline{\omega}$ and $\mathfrak{D}_\nu^J \nabla^P \omega$, see remark a) of Theorem 13.1.

Let us start with

$$\{\mathfrak{D}_\nu^J \nabla^P \text{tr}\chi\}$$

We start trying to control $\mathfrak{D}_\nu^J \nabla^P \text{tr}\chi$ for (J, P) such that $(J + P) = N$. Therefore we make the following assumptions for $(\tilde{J}, \tilde{P}) = (J - k, P + k)$ with $k \geq 1$ and all couples (\tilde{J}, \tilde{P}) such that $(\tilde{J} + \tilde{P}) < N$.¹⁰⁹

$$\begin{aligned} |r^{1+\tilde{J}+\tilde{P}+\sigma(\tilde{P})-\frac{2}{p}} \mathfrak{D}_\nu^{\tilde{J}} \nabla^{\tilde{P}} \text{tr}\chi|_{p,S} &\leq F \left(\frac{(\tilde{J} + \tilde{P})!}{(\tilde{J} + \tilde{P})^\alpha} \frac{e^{(\tilde{J}+\tilde{P}-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^{\tilde{J}+\tilde{P}}} \right) \\ |r^{1+\tilde{J}+\tilde{P}+\sigma(\tilde{P})-\frac{2}{p}} \mathfrak{D}_\nu^{\tilde{J}} \nabla^{\tilde{P}} U|_{p,S} &\leq F \left(\frac{(\tilde{J} + \tilde{P})!}{(\tilde{J} + \tilde{P})^\alpha} \frac{e^{(\tilde{J}+\tilde{P}-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^{\tilde{J}+\tilde{P}}} \right) \\ |r^{2+\tilde{J}+\tilde{P}-\frac{2}{p}} \mathfrak{D}_\nu^{\tilde{J}} \nabla^{\tilde{P}} \hat{\chi}|_{p,S} &\leq F \left(\frac{(\tilde{J} + \tilde{P})!}{(\tilde{J} + \tilde{P})^\alpha} \frac{e^{(\tilde{J}+\tilde{P}-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^{\tilde{J}+\tilde{P}}} \right). \end{aligned} \quad (13.58)$$

Starting from the transport equation for $U = \Omega^{-1} \text{tr}\chi$

$$\mathfrak{D}_\nu U + \frac{\Omega \text{tr}\chi}{2} U + |\hat{\chi}|^2 = 0, \quad (13.59)$$

$$\begin{aligned} \mathfrak{D}_\nu^J \nabla^P U &= \mathfrak{D}_\nu^{J-1} \nabla^P \mathfrak{D}_\nu U + \mathfrak{D}_\nu^{J-1} [\mathfrak{D}_\nu, \nabla^P] U = \mathfrak{D}_\nu^{J-1} \nabla^P \left(-\Omega \frac{\text{tr}\chi}{2} U - |\hat{\chi}|^2 \right) + \mathfrak{D}_\nu^{J-1} [\mathfrak{D}_\nu, \nabla^P] U \\ &= -\frac{1}{2} \mathfrak{D}_\nu^{J-1} \nabla^P (\Omega \text{tr}\chi U) - \mathfrak{D}_\nu^{J-1} \nabla^P |\hat{\chi}|^2 + \mathfrak{D}_\nu^{J-1} [\mathfrak{D}_\nu, \nabla^P] U \\ &= -\frac{1}{2} \sum_{q=0}^{J-1} \sum_{h=0}^P \binom{J-1}{q} \binom{P}{h} (\mathfrak{D}_\nu^q \nabla^h \Omega \text{tr}\chi) (\mathfrak{D}_\nu^{J-1-q} \nabla^{P-h} U) - \sum_{q=0}^{J-1} \sum_{h=0}^P \binom{J-1}{q} \binom{P}{h} (\mathfrak{D}_\nu^q \nabla^h \hat{\chi}) \cdot (\mathfrak{D}_\nu^{J-1-q} \nabla^{P-h} \hat{\chi}) \\ &\quad + \mathfrak{D}_\nu^{J-1} ([\mathfrak{D}_\nu, \nabla^P] U). \end{aligned} \quad (13.60)$$

Recall that the following relation holds

$$[\nabla^P, \frac{\mathcal{D}}{\partial \nu}] f = \sum_{k=0}^{P-1} \nabla^k [\nabla, \frac{\mathcal{D}}{\partial \nu}] \nabla^{P-k-1} f \quad (13.61)$$

and from equation 9.1

$$[\nabla_\mu, \frac{\mathcal{D}}{\partial \nu}] U_{\nu_1 \dots \nu_k} = - \sum_{j=1}^k C_{\mu \nu_j}^{\sigma_j} U_{\nu_1 \dots \sigma_j \dots \nu_k} + \Omega \chi_\mu^\rho (\nabla_\rho U)_{\nu_1 \dots \nu_k},$$

¹⁰⁹We use for notational simplicity an overall constant $F > F_i$.

it follows

$$\begin{aligned}
[\nabla^P, \frac{\mathcal{D}}{\partial \nu}]f &= \sum_{k=0}^{P-1} \nabla^k \left(-(P-k-1)C\nabla^{P-k-1}f + \Omega\chi\nabla^{P-k}f \right) \\
&= - \sum_{k=0}^{P-1} (P-k-1) \sum_{J=0}^k \binom{k}{J} (\nabla^J C) \nabla^{P-1-J}f + \sum_{k=0}^{P-1} \sum_{J=0}^k \binom{k}{J} (\nabla^J \Omega\chi) \nabla^{P-J}f .
\end{aligned} \tag{13.62}$$

As the following relation holds

$$\sum_{k=0}^{P-1} (P-k-1) \sum_{j=0}^k \binom{k}{j} = \sum_{j=0}^{P-1} \sum_{k=j}^{P-1} (P-k-1) \binom{k}{j} = \sum_{j=0}^{P-2} \binom{P}{P-2-j} = \sum_{l=0}^{P-2} \binom{P}{l} \tag{13.63}$$

we can write

$$\begin{aligned}
[\frac{\mathcal{D}}{\partial \nu}, \nabla^P]f &= \sum_{k=0}^{P-1} \nabla^k \left((P-k-1)C\nabla^{P-k-1}f - \Omega\chi\nabla^{P-k}f \right) \\
&= \sum_{j=0}^{P-2} \binom{P}{P-2-j} (\nabla^j C) \nabla^{P-1-j}f - \sum_{k=0}^{P-1} \sum_{j=0}^k \binom{k}{j} (\nabla^j \Omega\chi) \nabla^{P-j}f .
\end{aligned} \tag{13.64}$$

Therefore we can write

$$\begin{aligned}
\mathfrak{D}_\nu^J \nabla^P U &= \\
&- \frac{1}{2} \sum_{q=0}^{J-1} \sum_{h=0}^P \binom{J-1}{q} \binom{P}{h} (\mathfrak{D}_\nu^q \nabla^h \Omega \text{tr} \chi) (\mathfrak{D}_\nu^{J-1-q} \nabla^{P-h} U) \\
&- \sum_{q=0}^{J-1} \sum_{h=0}^P \binom{J-1}{q} \binom{P}{h} (\mathfrak{D}_\nu^q \nabla^h \hat{\chi}) \cdot (\mathfrak{D}_\nu^{J-1-q} \nabla^{P-h} \hat{\chi}) \\
&+ \sum_{q=0}^{J-1} \binom{J-1}{q} \sum_{k=0}^{P-1} (P-k-1) \sum_{j=0}^k \binom{k}{j} (\mathfrak{D}_\nu^q \nabla^j C) \mathfrak{D}_\nu^{J-1-q} \nabla^{P-1-j} U \\
&- \sum_{q=0}^{J-1} \sum_{k=0}^{P-1} \sum_{j=0}^k \binom{J-1}{q} \binom{k}{j} (\mathfrak{D}_\nu^q \nabla^j \Omega\chi) \mathfrak{D}_\nu^{J-1-q} \nabla^{P-j} U .
\end{aligned} \tag{13.65}$$

We start estimating the last sum which turns out to be the more delicate and which we rewrite as

$$\begin{aligned}
&\sum_{q=0}^{J-1} \sum_{k=0}^{P-1} \sum_{j=0}^k \binom{J-1}{q} \binom{k}{j} (\mathfrak{D}_\nu^q \nabla^j \Omega\chi) \mathfrak{D}_\nu^{J-1-q} \nabla^{P-j} U \\
&= \sum_{q=0}^{J-1} \sum_{h=0}^{P-1} \binom{J-1}{q} \binom{P}{h+1} (\mathfrak{D}_\nu^q \nabla^h \Omega\chi) \mathfrak{D}_\nu^{J-1-q} \nabla^{P-h} U .
\end{aligned} \tag{13.66}$$

In fact the more delicate part of this last sum is the one with $\text{tr}\chi$, part of χ , omitting Ω . Therefore we look at

$$\sum_{q=0}^{J-1} \sum_{h=0}^{P-1} \binom{J-1}{q} \binom{P}{h+1} (\mathfrak{D}_\nu^q \nabla^h \text{tr}\chi) (\mathfrak{D}_\nu^{J-1-q} \nabla^{P-h} U) \quad (13.67)$$

We observe, preliminary, that, denoting $N = J + P$,

$$\begin{aligned} \nabla^N AB &= \sum_{k=0}^N \binom{N}{k} \nabla^k A \nabla^{N-k} B = \nabla^J \nabla^P AB = \sum_{q=0}^J \sum_{h=0}^P \binom{J}{q} \binom{P}{h} \nabla^{q+h} A \nabla^{N-(q+h)} B \\ &= \sum_{k=0}^N \left(\sum_{q=0}^J \sum_{h=0}^P \delta(q+h=k) \binom{J}{q} \binom{P}{h} \right) \nabla^k A \nabla^{N-k} B \end{aligned} \quad (13.68)$$

therefore for any couple (J, P) such that $J + P = N$ the following relation holds which we use later on,

$$\begin{aligned} \binom{N}{k} &= \left(\sum_{q=0}^J \sum_{h=0}^P \delta(q+h=k) \binom{J}{q} \binom{P}{h} \right) \\ \sum_{k=0}^N \binom{N}{k} &= \sum_{k=0}^N \left(\sum_{q=0}^J \sum_{h=0}^P \delta(q+h=k) \binom{J}{q} \binom{P}{h} \right) \\ \sum_{q=0}^J \sum_{h=0}^P \binom{J}{q} \binom{P}{h} &= \sum_{q=0}^J \sum_{h=0}^P \binom{J}{q} \binom{P}{h} \cdot \end{aligned}$$

Therefore in conclusion

$$\sum_{k=0}^N \binom{N}{k} = \sum_{q=0}^J \sum_{h=0}^P \binom{J}{q} \binom{P}{h} . \quad (13.69)$$

We have the following estimate, recalling that $\sigma(0) = 0$, $\sigma(P) = 1$, for $P \geq 1$,

$$\begin{aligned} &\sum_{q=0}^{J-1} \sum_{h=0}^{P-1} \binom{J-1}{q} \binom{P}{h+1} \left| r^{1+J+P+\sigma(P)-\frac{2}{p}} (\mathfrak{D}_\nu^q \nabla^h \text{tr}\chi) (\mathfrak{D}_\nu^{J-1-q} \nabla^{P-h} U) \right|_{p,S} \\ &\leq \sum_{q=0}^{J-1} \sum_{h=0}^{P-1} \chi(J+P=N) \binom{J-1}{q} \binom{P}{h+1} \left| r^{1+J+P+\sigma(P)-\frac{2}{p}} (\mathfrak{D}_\nu^q \nabla^h \text{tr}\chi) (\mathfrak{D}_\nu^{J-1-q} \nabla^{P-h} U) \right|_{p,S} \\ &\leq \sum_{q=0}^{J-1} \sum_{h=0}^{P-1} \chi \left(J+P=N ; q+h \leq \left\lfloor \frac{N-1}{2} \right\rfloor \right) \binom{J-1}{q} \binom{P}{h+1} \left| r^{1+J+P+\sigma(P)-\frac{2}{p}} (\mathfrak{D}_\nu^q \nabla^h \text{tr}\chi) (\mathfrak{D}_\nu^{J-1-q} \nabla^{P-h} U) \right|_{p,S} \\ &+ \sum_{q=0}^{J-1} \sum_{h=0}^{P-1} \chi \left(J+P=N ; q+h \geq \left\lfloor \frac{N-1}{2} \right\rfloor + 1 \right) \binom{J-1}{q} \binom{P}{h+1} \left| r^{1+J+P+\sigma(P)-\frac{2}{p}} (\mathfrak{D}_\nu^q \nabla^h \text{tr}\chi) (\mathfrak{D}_\nu^{J-1-q} \nabla^{P-h} U) \right|_{p,S} . \end{aligned} \quad (13.70)$$

Let us consider the first sum

$$\begin{aligned}
& \sum_{q=0}^{J-1} \sum_{h=0}^{P-1} \chi \left(q+h \leq \left\lfloor \frac{N-1}{2} \right\rfloor \right) \binom{J-1}{q} \binom{P}{h+1} \left| r^{1+J+P+\sigma(P)-\frac{2}{p}} (\mathfrak{P}_\nu^q \nabla^h \text{tr} \chi) (\mathfrak{P}_\nu^{J-1-q} \nabla^{P-h} U) \right|_{p,S} \\
& \leq \sum_{q=0}^{J-1} \sum_{h=0}^{P-1} \chi \left(q+h \leq \left\lfloor \frac{N-1}{2} \right\rfloor \right) \binom{J-1}{q} \binom{P}{h+1} \left| r^{1+q+h+\sigma(h)} (\mathfrak{P}_\nu^q \nabla^h \text{tr} \chi) \right|_{\infty,S} \left| r^{J+P-q-h+\sigma(P-h)-\frac{2}{p}} (\mathfrak{P}_\nu^{J-1-q} \nabla^{P-h} U) \right|_{p,S} \\
& \leq \left(F \frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta+\Gamma_0(\lambda))}}{\rho_0^N} \right) \left(F e^{-2(\delta_0+\Gamma_0(\lambda))} \right) \cdot \\
& \sum_{q=0}^{J-1} \sum_{h=0}^{P-1} \chi \left(q+h \leq \left\lfloor \frac{N-1}{2} \right\rfloor \right) \binom{J-1}{q} \binom{P}{h+1} \left(\frac{(q+h+1)!}{((q+h+1)^\alpha)} \right) \left(\frac{(J-1+P-(h+q))!}{(J-1+P-(h+q))^\alpha} \right) \frac{N^\alpha}{N!} . \tag{13.71}
\end{aligned}$$

Denoting $\tilde{h} = h+1$

$$\begin{aligned}
& \leq \left(F \frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta+\Gamma_0(\lambda))}}{\rho_0^N} \right) \left(F e^{-2(\delta_0+\Gamma_0(\lambda))} \right) \cdot \\
& \sum_{q=0}^{J-1} \sum_{\tilde{h}=1}^P \chi \left(q+\tilde{h} \leq \left\lfloor \frac{N-1}{2} \right\rfloor + 1 \right) \binom{J-1}{q} \binom{P}{\tilde{h}} \left(\frac{(q+\tilde{h})!}{(q+\tilde{h})^\alpha} \right) \left(\frac{(J+P-(q+\tilde{h}))!}{(J+P-(q+\tilde{h}))^\alpha} \right) \frac{N^\alpha}{N!} \tag{13.72} \\
& \leq \left(F \frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta+\Gamma_0(\lambda))}}{\rho_0^N} \right) \left(f e^{-2(\delta+\Gamma_0(\lambda))} \right) \cdot \\
& \sum_{q=0}^{J-1} \sum_{\tilde{h}=0}^P \chi \left(q+\tilde{h} \leq \left\lfloor \frac{N-1}{2} \right\rfloor + 1 \right) \binom{J-1}{q} \binom{P}{\tilde{h}} \left(\frac{(q+\tilde{h})!}{((q+\tilde{h})^\alpha)} \right) \left(\frac{(J+P-(q+\tilde{h}))!}{(J+P-(q+\tilde{h}))^\alpha} \right) \frac{N^\alpha}{N!} ,
\end{aligned}$$

using the previous relation 13.69, with $J+P=N$, denoting $k=q+\tilde{h}$

$$\begin{aligned}
& \sum_{k=0}^N \binom{N}{k} = \sum_{q=0}^J \sum_{h=0}^P \binom{J}{q} \binom{P}{h} , \\
& \leq \cdot \left(F \frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^N} \right) \left(F e^{-2(\delta_0+\Gamma_0(\lambda))} \right) \cdot \\
& \sum_{q=0}^{J-1} \sum_{\tilde{h}=0}^P \chi \left(k=q+\tilde{h} \leq \left\lfloor \frac{N-1}{2} \right\rfloor + 1 \right) \binom{J-1}{q} \binom{P}{\tilde{h}} \left(\frac{k!}{k^\alpha} \right) \left(\frac{(N-k)!}{(N-k)^\alpha} \right) \frac{N^\alpha}{N!} \\
& \leq \left(F \frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^N} \right) \left(F e^{-2(\delta_0+\Gamma_0(\lambda))} \right) \cdot \sum_{k=0}^{\left\lfloor \frac{N-1}{2} \right\rfloor + 1} \binom{N-1}{k} \left(\frac{k!}{k^\alpha} \right) \left(\frac{(N-k)!}{(N-k)^\alpha} \right) \frac{N^\alpha}{N!} \\
& \leq \left(F \frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^N} \right) \left(F e^{-2(\delta_0+\Gamma_0(\lambda))} \right) \cdot \sum_{k=0}^{\left\lfloor \frac{N-1}{2} \right\rfloor + 1} \frac{(N-1)!}{(N-1-k)!} \left(\frac{1}{k^\alpha} \right) \left(\frac{(N-k)!}{(N-k)^\alpha} \right) \frac{N^\alpha}{N!}
\end{aligned}$$

$$\begin{aligned}
&\leq \left(F \frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_0^N} \right) \left(F e^{-2(\delta_0 + \Gamma_0(\lambda))} \right) \cdot \sum_{k=0}^{\left[\frac{N-1}{2}\right]+1} \left(\frac{1}{k^\alpha} \right) \left(\frac{(N-k)}{(N-k)^\alpha} \right) \frac{N^\alpha}{N} \\
&\leq \left(F \frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_0^N} \right) \left(F e^{-2(\delta_0 + \Gamma_0(\lambda))} \right) \cdot \sum_{k=0}^{\left[\frac{N-1}{2}\right]+1} \left(\frac{1}{k^\alpha} \right) \left(\frac{N^\alpha}{(N-k)^\alpha} \right) \\
&\leq \left(F \frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_0^N} \right) , \tag{13.73}
\end{aligned}$$

provided that $\alpha > 1$ and δ sufficiently large such that,

$$\left(F e^{-2(\delta_0 + \Gamma_0(\lambda))} \right) \cdot \sum_{k=0}^{\left[\frac{N-1}{2}\right]+1} \left(\frac{1}{k^\alpha} \right) \left(\frac{N^\alpha}{(N-k)^\alpha} \right) << 1 . \tag{13.74}$$

Let us consider now the second sum

$$\begin{aligned}
&\sum_{q=0}^{J-1} \sum_{h=0}^{P-1} \chi \left(q+h \geq \left\lfloor \frac{N-1}{2} \right\rfloor + 1 \right) \binom{J-1}{q} \binom{P}{h+1} \left| r^{1+J+P+\sigma(P)-\frac{2}{p}} (\mathfrak{P}_4^q \nabla^h \text{tr} \chi) (\mathfrak{P}_4^{J-1-q} \nabla^{P-h} U) \right|_{p,S} \\
&\leq \sum_{q=0}^{J-1} \sum_{h=0}^{P-1} \chi \left(q+h \geq \left\lfloor \frac{N-1}{2} \right\rfloor + 1 \right) \binom{J-1}{q} \binom{P}{h+1} \left| r^{1+q+h+\sigma(h)-\frac{2}{p}} (\mathfrak{P}_4^q \nabla^h \text{tr} \chi) \right|_{p,S} \left| r^{J+P-q-h+\sigma(P-h)} (\mathfrak{P}_4^{J-1-q} \nabla^{P-h} U) \right|_{\infty,S} \\
&\leq \left(F \frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_2^N} \right) \left(F e^{-2(\delta_0 + \Gamma_0(\lambda))} \right) . \\
&\sum_{q=0}^{J-1} \sum_{h=0}^{P-1} \chi \left(q+h \geq \left\lfloor \frac{N-1}{2} \right\rfloor + 1 \right) \binom{J-1}{q} \binom{P}{h+1} \left(\frac{(q+h)!}{((q+h)^\alpha)} \right) \left(\frac{(J+P+1-(h+q))!}{(J+P+1-(h+q))^\alpha} \right) \frac{N^\alpha}{N!} . \tag{13.75}
\end{aligned}$$

$$\begin{aligned}
&\leq \left(F \frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_0^N} \right) \left(F e^{-2(\delta_0 + \Gamma_0(\lambda))} \right) . \\
&\sum_{q=0}^{J-1} \sum_{h=0}^{P-1} \chi \left(q+h \geq \left\lfloor \frac{N-1}{2} \right\rfloor + 1 \right) \binom{J-1}{q} \binom{P}{h} \left(\frac{(q+h)!}{((q+h)^\alpha)} \right) \left(\frac{(J+P+1-(q+h))!}{(J+P+1-(q+h))^\alpha} \right) \frac{N^\alpha}{N!} . \tag{13.76}
\end{aligned}$$

Using the previous relation 13.69, with $J+P=N$,

$$\sum_{k=0}^N \binom{N}{k} = \sum_{q=0}^J \sum_{h=0}^P \binom{J}{q} \binom{P}{h} ,$$

denoting $k = q + h$

$$\leq \left(F \frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_0^N} \right) \left(F e^{-2(\delta_0 + \Gamma_0(\lambda))} \right) \cdot \sum_{k=\left[\frac{N-1}{2}\right]+1}^{N-1} \binom{N-1}{k} \left(\frac{k!}{k^\alpha} \right) \left(\frac{(N+1-k)!}{(N+1-k)^\alpha} \right) \frac{N^\alpha}{N!}$$

$$\begin{aligned}
&\leq \left(F \frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_0^N} \right) \left(F e^{-2(\delta_0 + \Gamma_0(\lambda))} \right) \cdot \sum_{k=\lfloor \frac{N-1}{2} \rfloor + 1}^{N-1} \frac{(N-1)!}{k!(N-1-k)!} \left(\frac{k!}{k^\alpha} \right) \left(\frac{(N+1-k)!}{(N+1-k)^\alpha} \right) \frac{N^\alpha}{N!} \\
&\leq \left(F \frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_0^N} \right) \left(F e^{-2(\delta_0 + \Gamma_0(\lambda))} \right) \cdot \sum_{k=\lfloor \frac{N-1}{2} \rfloor + 1}^{N-1} \left(\frac{1}{k^\alpha} \right) \left(\frac{(N+1-k)(N-k)}{(N+1-k)^\alpha} \right) \frac{N^\alpha}{N} \\
&\leq \left(F \frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_0^N} \right) \left(F e^{-2(\delta_0 + \Gamma_0(\lambda))} \right) \cdot \sum_{k=\lfloor \frac{N-1}{2} \rfloor + 1}^{N-1} \left(\frac{1}{k^\alpha} \right) \left(\frac{(N+1-k)(N-k)}{(N+1-k)^\alpha} \right) \frac{N^\alpha}{N} \\
&\leq \left(F \frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_0^N} \right) \left(F e^{-2(\delta_0 + \Gamma_0(\lambda))} \right) \cdot 2^\alpha \sum_{k=\lfloor \frac{N-1}{2} \rfloor + 2}^{N-1} \frac{1}{(N+1-k)^{\alpha-1}} \\
&\leq \left(F \frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_0^N} \right) (c 2^\alpha F e^{-2(\delta_0 + \Gamma_0(\lambda))}) \leq \left(F \frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_0^N} \right), \tag{13.77}
\end{aligned}$$

choosing δ sufficiently large.

To complete these estimates we look at the less dangerous part of the commutator part, let us recall first the complete expression of the commutator, see 13.65

$$\begin{aligned}
\mathfrak{D}_\nu^{J-1}[\mathfrak{D}_\nu, \nabla^P]U &= \sum_{q=0}^{J-1} \binom{J-1}{q} \sum_{h=0}^{P-1} (P-h-1) \sum_{j=0}^h \binom{h}{j} (\mathfrak{D}_\nu^q \nabla^j C) \mathfrak{D}_\nu^{J-1-q} \nabla^{P-1-j} U \\
&\quad - \sum_{q=0}^{J-1} \sum_{h=0}^{P-1} \sum_{j=0}^h \binom{J-1}{q} \binom{h}{j} (\mathfrak{D}_\nu^q \nabla^j \chi) \mathfrak{D}_\nu^{J-1-q} \nabla^{P-j} U. \tag{13.78}
\end{aligned}$$

As the following relation holds

$$\sum_{h=0}^{P-1} (P-h-1) \sum_{j=0}^h \binom{h}{j} = \sum_{j=0}^{P-1} \sum_{h=j}^{P-1} (P-h-1) \binom{h}{j} = \sum_{j=0}^{P-2} \binom{P}{P-2-j} = \sum_{l=0}^{P-2} \binom{P}{l} \tag{13.79}$$

we can write

$$\begin{aligned}
\mathfrak{D}_\nu^{J-1}[\mathfrak{D}_\nu, \nabla^P]U &= \sum_{q=0}^{J-1} \sum_{j=0}^{P-2} \binom{J-1}{q} \binom{P}{P-2-j} (\mathfrak{D}_\nu^q \nabla^j C) \mathfrak{D}_\nu^{J-1-q} \nabla^{P-1-j} U \\
&\quad - \sum_{q=0}^{J-1} \sum_{h=0}^{P-1} \sum_{j=0}^h \binom{J-1}{q} \binom{h}{j} (\mathfrak{D}_\nu^q \nabla^j \chi) \mathfrak{D}_\nu^{J-1-q} \nabla^{P-j} U \\
&= \sum_{q=0}^{J-1} \sum_{h=0}^{P-2} \binom{J-1}{q} \binom{P}{h} (\mathfrak{D}_\nu^q \nabla^h C) \mathfrak{D}_\nu^{J-1-q} \nabla^{P-h-1} U \\
&\quad - \sum_{q=0}^{J-1} \sum_{h=0}^{P-1} \binom{J-1}{q} \binom{P}{h} (\mathfrak{D}_\nu^q \nabla^h \chi) (\mathfrak{D}_\nu^{J-1-q} \nabla^{P-h-1} U) \tag{13.80}
\end{aligned}$$

The terms in the second line have already been estimated, the term in the first line has exactly the same structure and therefore produces the same estimate. The first two lines of $\mathfrak{D}_\nu^J \nabla^P U$, 13.65, can be estimated in the same way and they are even easier to bound,¹¹⁰ therefore we do not report their estimates.

Remarks:

- i) Observe that in the estimates of the $|\cdot|_{\infty, S}$ norms in terms of the $|\cdot|_{p=4, S}$ norms with an extra ∇ we have neglected the position of this extra ∇ ; it is easy to see that taking it into account does not change the final estimate (at most we have to choose a large α).
- ii) As already said, notice that in the case of the initial data we do not need the energy estimates due to the fact that all the mixed derivatives of $\hat{\chi}$ are already at our disposal. This is crucial as in the case of the initial data we do not have any possibility to exploit the energy estimates to bound the null Riemann components.

Let us consider the structure equation,

$$\mathfrak{D}_4 \zeta + \zeta \chi + \text{tr} \chi \zeta - \text{div} \chi + \nabla \text{tr} \chi + \mathfrak{D}_4 \nabla \log \Omega = 0 .$$

All the terms can be treated in the same way as before and we do not repeat here, therefore we can prove the estimates, with $J + P = N$,

Moreover, by the identity

$$\eta = \zeta + \nabla \log \Omega,$$

see equations 3.1.33 [Kl-Ni2], and the assumptions on Ω ,

we obtain

$$\begin{aligned} |r^{2+J+P-\frac{2}{p}} \mathfrak{D}_\nu^J \nabla^P \eta|_{p, S} &\leq F \left(\frac{(J+P)!}{(J+P)^\alpha} \frac{e^{((J+P)-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_0^{J+P}} \right) \\ |r^{2+J+P-\frac{2}{p}} \mathfrak{D}_\nu^J \nabla^P \zeta|_{p, S} &\leq F \left(\frac{(J+P+1)!}{(J+P+1)^\alpha} \frac{e^{((J+P)-1)(\delta_0 + \Gamma_0(\lambda))}}{\rho_0^{J+P+1}} \right) . \end{aligned} \quad (13.81)$$

13.4.1 Estimate of the underlined quantities

As already discussed we can use also the transport equations implying a loss of derivatives as the angular derivatives estimates are already under control.

$\{\mathfrak{D}_\nu^J \nabla^P \text{tr} \underline{\chi}\}$:

In this case we use the transport equation for $\text{tr} \underline{\chi}$ along the outgoing cones which we write, denoting $\underline{U} = \Omega \text{tr} \underline{\chi}$,

$$\mathfrak{D}_\nu \underline{U} + \frac{1}{2} \Omega \text{tr} \chi \underline{U} + \Omega^2 \hat{\chi} \cdot \underline{\hat{\chi}} - 2\Omega^2 \text{div} \underline{\eta} - 2\Omega^2 |\underline{\eta}|^2 - 2\Omega^2 \rho = 0 . \quad (13.82)$$

¹¹⁰Due to the fact that, comparing the first and the fourth line of 13.65 a factor $\binom{P}{h}$ is present, in the first one, instead of $\binom{P}{h+1}$.

Proceeding as before, we write

$$\begin{aligned}
\mathfrak{P}_\nu^J \nabla^P \underline{U} &= \mathfrak{P}_\nu^{J-1} \nabla^P (\mathfrak{P}_\nu \underline{U} + \mathfrak{P}_\nu^{J-1} [\mathfrak{P}_\nu, \nabla^P] \underline{U}) \\
&= -\mathfrak{P}_\nu^{J-1} \nabla^P \left(\frac{1}{2} \Omega \operatorname{tr} \chi \underline{U} + \Omega^2 \hat{\chi} \cdot \hat{\chi} - 2\Omega^2 \mathfrak{d} \mathfrak{v} \underline{\eta} - 2\Omega^2 |\underline{\eta}|^2 - 2\Omega^2 \rho \right) + \mathfrak{P}_\nu^{J-1} [\mathfrak{P}_\nu, \nabla^P] \underline{U} \\
&= -\mathfrak{P}_\nu^{J-1} \nabla^P \left(\frac{1}{2} \Omega \operatorname{tr} \chi \underline{U} + \Omega^2 \hat{\chi} \cdot \hat{\chi} - 2\Omega^2 \mathfrak{d} \mathfrak{v} \underline{\eta} - 2\Omega^2 |\underline{\eta}|^2 \right) + 2\mathfrak{P}_\nu^{J-1} \nabla^P (\Omega^2 \rho) \\
&\quad + \sum_{q=0}^{J-1} \sum_{j=0}^{P-2} \binom{J-1}{q} \binom{P}{h+1} (\mathfrak{P}_\nu^q \nabla^{(h-1)} C) \mathfrak{P}_\nu^{J-1-q} \nabla^{P-h} \hat{U} \\
&\quad - \sum_{q=0}^{J-1} \sum_{h=0}^{P-1} \binom{J-1}{q} \binom{P}{h+1} (\mathfrak{P}_\nu^q \nabla^h \chi) (\mathfrak{P}_\nu^{J-1-q} \nabla^{P-h} \hat{U})
\end{aligned}$$

It is immediate to realize that all the estimates can be done as the previous ones and, therefore, the final result is,

$$|r^{2+J+P-\frac{2}{p}} \mathfrak{P}_\nu^J \nabla^P \underline{U}|_{p,S} \leq F \left(\frac{(J+P)!}{(J+P)^\alpha} \frac{e^{((J+P)-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_0^{J+P}} \right). \quad (13.83)$$

$\{\mathfrak{P}_\nu^J \nabla^P \hat{\chi}\} :$

In this case we use the transport equation

$$\mathfrak{P}_\nu \hat{\chi} + \frac{\Omega}{2} \operatorname{tr} \chi \hat{\chi} + \frac{\Omega}{2} \operatorname{tr} \chi \hat{\chi} - 2\Omega \omega \hat{\chi} - \nabla \hat{\otimes} \underline{\eta} - \underline{\eta} \hat{\otimes} \underline{\eta} = 0$$

which we rewrite, denoting $\hat{\underline{U}} = \Omega \hat{\chi}$,

$$\mathfrak{P}_\nu \hat{\underline{U}} + \frac{\Omega}{2} \operatorname{tr} \chi \hat{\underline{U}} + \frac{\Omega}{2} \operatorname{tr} \chi (\Omega \hat{\chi}) - \Omega \nabla \hat{\otimes} \underline{\eta} - \Omega \underline{\eta} \hat{\otimes} \underline{\eta} = 0.$$

Once again

$$\begin{aligned}
\mathfrak{P}_\nu^J \nabla^P \hat{\underline{U}} &= \mathfrak{P}_\nu^{J-1} \nabla^P (\mathfrak{P}_\nu \hat{\underline{U}} + \mathfrak{P}_\nu^{J-1} [\mathfrak{P}_\nu, \nabla^P] \hat{\underline{U}}) \\
&= -\mathfrak{P}_\nu^{J-1} \nabla^P \left(\frac{\Omega}{2} \operatorname{tr} \chi \hat{\underline{U}} + \frac{\Omega}{2} \operatorname{tr} \chi (\Omega \hat{\chi}) - \Omega \nabla \hat{\otimes} \underline{\eta} - \Omega \underline{\eta} \hat{\otimes} \underline{\eta} \right) + \mathfrak{P}_\nu^{J-1} [\mathfrak{P}_\nu, \nabla^P] \hat{\underline{U}} \\
&= -\mathfrak{P}_\nu^{J-1} \nabla^P \left(\frac{\Omega}{2} \operatorname{tr} \chi \hat{\underline{U}} + \frac{\Omega}{2} \operatorname{tr} \chi (\Omega \hat{\chi}) - \Omega \nabla \hat{\otimes} \underline{\eta} - \Omega \underline{\eta} \hat{\otimes} \underline{\eta} \right) + \mathfrak{P}_\nu^{J-1} [\mathfrak{P}_\nu, \nabla^P] \hat{\underline{U}} \\
&\quad + \sum_{q=0}^{J-1} \sum_{j=0}^{P-2} \binom{J-1}{q} \binom{P}{h} (\mathfrak{P}_\nu^q \nabla^h C) \mathfrak{P}_\nu^{J-1-q} \nabla^{P-h} \hat{\underline{U}} \\
&\quad - \sum_{q=0}^{J-1} \sum_{h=0}^{P-1} \binom{J-1}{q} \binom{P}{h} (\mathfrak{P}_\nu^q \nabla^h \chi) (\mathfrak{P}_\nu^{J-1-q} \nabla^{P-h} \hat{\underline{U}})
\end{aligned}$$

and the inductive estimates are proved exactly in the same way obtaining as final result,

$$||\lambda|r^{1+J+P-\frac{2}{p}}\mathfrak{D}_\nu^J\mathfrak{V}^P\hat{U}|_{p,S} \leq F\left(\frac{(J+P)!}{(J+P)^\alpha}\frac{e^{((J+P)-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_0^{J+P}}\right). \quad (13.84)$$

Proceeding exactly in the same way we obtain the analogous estimates for the mixed derivatives with \mathfrak{D}_λ instead of \mathfrak{D}_ν .

14 Appendix to Section 9

Some commutation relations

The following relations hold:

$$\begin{aligned} & [\nabla_\mu, \frac{\mathcal{D}}{\partial\nu}]U_{\nu_1\dots\nu_k} = \\ & = -\Omega \sum_{j=1}^k \left[(\chi_{\mu\nu_j}\eta^{\sigma_j} - \chi_\mu^{\sigma_j}\eta_{\nu_j}) + \theta_\mu^C\theta_{\nu_j}^D R^{\sigma_j}(\cdot, e_C, e_4, e_D) \right] U_{\nu_1\dots\sigma_j\dots\nu_k} + \Omega\chi_\mu^\rho(\nabla_\rho U)_{\nu_1\dots\nu_k} \\ & = -\sum_{j=1}^k C_{\mu\nu_j}^{\sigma_j} U_{\nu_1\dots\sigma_j\dots\nu_k} + \Omega\chi_\mu^\rho(\nabla_\rho U)_{\nu_1\dots\nu_k}, \end{aligned} \quad (14.1)$$

$$\text{with} \quad C_{\mu\nu}^\sigma = \Omega \left[(\chi_{\mu\nu}\eta^\sigma - \chi_\mu^\sigma\eta_\nu) + \theta_\mu^C\theta_\nu^D R^\sigma(\cdot, e_C, e_4, e_D) \right]. \quad (14.2)$$

If U is a 1-form we have,

$$\begin{aligned} & [\nabla_\mu, \frac{\mathcal{D}}{\partial\nu}]U_\nu = \left[(\chi_{\mu\nu}\eta^\sigma - \chi_\mu^\sigma\eta_\nu) + \theta_\mu^C\theta_\nu^D R^\sigma(\cdot, e_C, e_4, e_D) \right] U_\sigma + \Omega\chi_\mu^\rho(\nabla_\rho U)_\nu \\ & = -C_{\mu\nu}^\sigma U_\sigma + \Omega\chi_\mu^\rho(\nabla_\rho U)_\nu \end{aligned} \quad (14.3)$$

We rewrite symbolically the last formula as:

$$[\nabla_\mu, \frac{\mathcal{D}}{\partial\nu}]U = -CU + D(\nabla U) \quad (14.4)$$

Where with C we mean a combination of null Riemann components and with F we mean a combination of connection coefficients and metric components. Similarly we have:

$$\begin{aligned} & [\nabla_\mu, \frac{\mathcal{D}}{\partial\lambda}]U_{\nu_1\dots\nu_k} = \\ & = -\Omega \sum_{j=1}^k \left[(\chi_{\mu\nu_j}\eta^{\sigma_j} - \chi_\mu^{\sigma_j}\eta_{\nu_j}) + \theta_\mu^C\theta_{\nu_j}^D R^{\sigma_j}(\cdot, e_C, e_3, e_D) \right] U_{\nu_1\dots\sigma_j\dots\nu_k} + \Omega\chi_\mu^\rho(\nabla_\rho U)_{\nu_1\dots\nu_k} \\ & = -\sum_{j=1}^k C_{\mu\nu_j}^{\sigma_j} U_{\nu_1\dots\sigma_j\dots\nu_k} + \Omega\chi_\mu^\rho(\nabla_\rho U)_{\nu_1\dots\nu_k}, \end{aligned} \quad (14.5)$$

which, in the case of U 1-form, becomes:

$$\begin{aligned} [\nabla_\mu, \frac{\mathcal{D}}{\partial\lambda}]U_\nu &= \left[(\chi_{\mu\nu_j} \eta^\sigma - \chi_\mu^\sigma \eta_\nu) + \theta_\mu^C \theta_\nu^D R^\sigma(\cdot, e_C, e_3, e_D) \right] U_\sigma + \Omega_{\underline{\chi}_\mu}^\rho (\nabla_\rho U)_\nu \\ &= -\underline{C}_{\mu\nu}^\sigma U_\sigma + \Omega_{\underline{\chi}_\mu}^\rho (\nabla_\rho U)_\nu \end{aligned} \quad (14.6)$$

Again we rewrite the last formula symbolically as

$$[\nabla_\mu, \frac{\mathcal{D}}{\partial\lambda}]U = -\underline{C}U + \underline{D}(\nabla U) . \quad (14.7)$$

The commutator of the tangential derivatives is,

$$\begin{aligned} [\nabla_\alpha, \nabla_\mu]U_{\nu_1\nu_2\dots\nu_q} &= \\ &= \left[(\theta_\mu^3 \underline{\chi}_\alpha^\beta + \theta_\mu^4 \chi_\alpha^\beta) - (\theta_\alpha^3 \underline{\chi}_\mu^\beta + \theta_\alpha^4 \chi_\mu^\beta) \right] \nabla_\beta U_{\nu_1\nu_2\dots\nu_q} - \theta_\alpha^C \theta_\mu^D \sum_{i=1}^q R^{\tilde{\nu}_i}(\cdot, e_C, \cdot, e_D)_{\nu_i} U_{\nu_1\nu_2\dots\tilde{\nu}_i\dots\nu_q} \end{aligned} \quad (14.8)$$

and when the commutator operates on a 2-form,

$$\begin{aligned} [\nabla_\alpha, \nabla_\mu]U_{\nu_1\nu_2} &= \left[(\theta_\mu^3 \underline{\chi}_\alpha^\beta + \theta_\mu^4 \chi_\alpha^\beta) - (\theta_\alpha^3 \underline{\chi}_\mu^\beta + \theta_\alpha^4 \chi_\mu^\beta) \right] \nabla_\beta U_{\nu_1\nu_2} \\ &\quad - \theta_\alpha^C \theta_\mu^D R^{\tilde{\nu}}(\cdot, e_C, \cdot, e_D)_{\nu_1} U_{\tilde{\nu}\nu_2} - \theta_\alpha^C \theta_\mu^D R^{\tilde{\nu}}(\cdot, e_C, \cdot, e_D)_{\nu_2} U_{\nu_1\tilde{\nu}} \end{aligned} \quad (14.9)$$

which we rewrite symbolically as

$$[\nabla_\mu, \frac{\mathcal{D}}{\partial\nu}]U = -\hat{C}U + \hat{F}(\nabla U) \quad (14.10)$$

Finally applying $[\nabla_\alpha, d\mathbf{v}]$ to a 2-form we have,

$$\begin{aligned} [\nabla_\alpha, d\mathbf{v}]U_{\nu_1\nu_2} &= [\nabla_\alpha, \nabla_\mu]U_\nu^\mu = \\ &= \left[(\theta_\mu^3 \underline{\chi}_\alpha^\beta + \theta_\mu^4 \chi_\alpha^\beta) - (\theta_\alpha^3 \underline{\chi}_\mu^\beta + \theta_\alpha^4 \chi_\mu^\beta) \right] \nabla_\beta U_\nu^\mu - 2\theta_\alpha^C \theta_\mu^D R^{\tilde{\nu}}(\cdot, e_C, \cdot, e_D)_\nu U_\nu^\mu , \end{aligned} \quad (14.11)$$

which we rewrite symbolically as

$$[\nabla_\alpha, d\mathbf{v}]U = -\overline{C}U + \overline{F}(\nabla U) \quad (14.12)$$

Notice that all the commutators have the same structure, namely a combination of products of connection coefficients and metric components time ∇U and a combination of null Riemann components times U .

Detailed derivation of all the previous relations:

Consider first the commutator $[\frac{\mathcal{D}}{\partial\nu}, \nabla_\mu]$ where with $\frac{\mathcal{D}}{\partial\nu}$ we denote the covariant derivative $\Omega\mathbf{P}_{e_4}$. Let $U_{\nu_1\dots\nu_k}$ be a covariant tensor tangent to S

$$[\frac{\mathcal{D}}{\partial\nu}, \nabla_\mu]U_{\nu_1\dots\nu_k} = \frac{\mathcal{D}}{\partial\nu}\nabla_\mu U_{\nu_1\dots\nu_k} - \nabla_\mu \frac{\mathcal{D}}{\partial\nu}U_{\nu_1\dots\nu_k}$$

and

$$\begin{aligned}\frac{\mathcal{D}}{\partial\nu}\nabla_\mu U_{\nu_1\dots\nu_k} &= \Omega\mathbf{P}_4\Pi_\mu^\rho\Pi_{\nu_1\dots\nu_k}^{\sigma_1\dots\sigma_k}D_\rho U_{\sigma_1\dots\sigma_k} \\ \nabla_\mu \frac{\mathcal{D}}{\partial\nu}U_{\nu_1\dots\nu_k} &= \Pi_\mu^\rho\Pi_{\nu_1\dots\nu_k}^{\lambda_1\dots\lambda_k}D_\rho\Omega\Pi_{\lambda_1\dots\lambda_k}^{\sigma_1\dots\sigma_k}e_4^\tau D_\tau U_{\sigma_1\dots\sigma_k}\end{aligned}\quad (14.13)$$

where we defined

$$\begin{aligned}\Pi_{\nu_1\dots\nu_k}^{\sigma_1\dots\sigma_k} &\equiv \Pi_{\nu_1}^{\sigma_1}\dots\Pi_{\nu_k}^{\sigma_k} \\ \Pi_{\nu_i}^{\sigma_i} &= \delta_{\nu_i}^{\sigma_i} - (e_3\theta_3 + e_4\theta_4)_{\nu_i}^{\sigma_i} = \delta_{\nu_i}^{\sigma_i} + \frac{1}{2}g_{\nu_j\rho}(e_3^{\sigma_i}e_4^\rho + e_4^{\sigma_i}e_3^\rho) .\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\mathcal{D}}{\partial\nu}\nabla_\mu U_{\nu_1\dots\nu_k} &= \Omega(\Pi_\mu^{\mu'}\mathbf{D}_4\Pi_{\mu'}^\rho)\Pi_{\nu_1\dots\nu_k}^{\sigma_1\dots\sigma_k}D_\rho U_{\sigma_1\dots\sigma_k} + \Omega\Pi_\mu^\rho(\Pi_{\nu_1\dots\nu_k}^{\nu'_1\dots\nu'_k}\mathbf{D}_4\Pi_{\nu'_1\dots\nu'_k}^{\sigma_1\dots\sigma_k})D_\rho U_{\sigma_1\dots\sigma_k} \\ &+ \Omega\Pi_\mu^\rho\Pi_{\nu_1\dots\nu_k}^{\sigma_1\dots\sigma_k}e_4^\tau D_\tau D_\rho U_{\sigma_1\dots\sigma_k} ,\end{aligned}\quad (14.14)$$

$$\begin{aligned}\nabla_\mu \frac{\mathcal{D}}{\partial\nu}U_{\nu_1\dots\nu_k} &= \Pi_\mu^\rho(D_\rho\Omega)\Pi_{\nu_1\dots\nu_k}^{\sigma_1\dots\sigma_k}e_4^\tau D_\tau U_{\sigma_1\dots\sigma_k} + \Omega\Pi_\mu^\rho(\Pi_{\nu_1\dots\nu_k}^{\lambda_1\dots\lambda_k}D_\rho\Pi_{\lambda_1\dots\lambda_k}^{\sigma_1\dots\sigma_k})e_4^\tau D_\tau U_{\sigma_1\dots\sigma_k} \\ &+ \Omega\Pi_\mu^\rho\Pi_{\nu_1\dots\nu_k}^{\sigma_1\dots\sigma_k}(D_\rho e_4^\tau)D_\tau U_{\sigma_1\dots\sigma_k} + \Omega\Pi_\mu^\rho\Pi_{\nu_1\dots\nu_k}^{\sigma_1\dots\sigma_k}e_4^\tau D_\rho D_\tau U_{\sigma_1\dots\sigma_k} .\end{aligned}\quad (14.15)$$

From it

$$\begin{aligned}[\nabla_\mu, \frac{\mathcal{D}}{\partial\nu}]U_{\nu_1\dots\nu_k} &= \Pi_\mu^\rho(D_\rho\Omega)\Pi_{\nu_1\dots\nu_k}^{\sigma_1\dots\sigma_k}e_4^\tau D_\tau U_{\sigma_1\dots\sigma_k} + \Omega\Pi_\mu^\rho(\Pi_{\nu_1\dots\nu_k}^{\lambda_1\dots\lambda_k}D_\rho\Pi_{\lambda_1\dots\lambda_k}^{\sigma_1\dots\sigma_k})e_4^\tau D_\tau U_{\sigma_1\dots\sigma_k} \\ &+ \Omega\Pi_\mu^\rho\Pi_{\nu_1\dots\nu_k}^{\sigma_1\dots\sigma_k}(D_\rho e_4^\tau)D_\tau U_{\sigma_1\dots\sigma_k} + \Omega\Pi_\mu^\rho\Pi_{\nu_1\dots\nu_k}^{\sigma_1\dots\sigma_k}e_4^\tau D_\rho D_\tau U_{\sigma_1\dots\sigma_k} \\ &- \Omega(\Pi_\mu^{\mu'}\mathbf{D}_4\Pi_{\mu'}^\rho)\Pi_{\nu_1\dots\nu_k}^{\sigma_1\dots\sigma_k}D_\rho U_{\sigma_1\dots\sigma_k} - \Omega\Pi_\mu^\rho(\Pi_{\nu_1\dots\nu_k}^{\nu'_1\dots\nu'_k}\mathbf{D}_4\Pi_{\nu'_1\dots\nu'_k}^{\sigma_1\dots\sigma_k})D_\rho U_{\sigma_1\dots\sigma_k} \\ &- \Omega\Pi_\mu^\rho\Pi_{\nu_1\dots\nu_k}^{\sigma_1\dots\sigma_k}e_4^\tau D_\tau D_\rho U_{\sigma_1\dots\sigma_k} \\ &= \Pi_\mu^\rho(D_\rho\log\Omega)\Pi_{\nu_1\dots\nu_k}^{\sigma_1\dots\sigma_k}\frac{\mathcal{D}}{\partial\nu}U_{\sigma_1\dots\sigma_k} + \Pi_\mu^\rho(\Pi_{\nu_1\dots\nu_k}^{\lambda_1\dots\lambda_k}D_\rho\Pi_{\lambda_1\dots\lambda_k}^{\sigma_1\dots\sigma_k})\frac{\mathcal{D}}{\partial\nu}U_{\sigma_1\dots\sigma_k} \\ &- \Omega(\Pi_\mu^{\mu'}\mathbf{D}_4\Pi_{\mu'}^\rho)\Pi_{\nu_1\dots\nu_k}^{\sigma_1\dots\sigma_k}D_\rho U_{\sigma_1\dots\sigma_k} - \Omega\Pi_\mu^\rho(\Pi_{\nu_1\dots\nu_k}^{\nu'_1\dots\nu'_k}\mathbf{D}_4\Pi_{\nu'_1\dots\nu'_k}^{\sigma_1\dots\sigma_k})D_\rho U_{\sigma_1\dots\sigma_k} \\ &+ \Omega\Pi_\mu^\rho\Pi_{\nu_1\dots\nu_k}^{\sigma_1\dots\sigma_k}(D_\rho e_4^\tau)D_\tau U_{\sigma_1\dots\sigma_k} + \Omega\Pi_\mu^\rho\Pi_{\nu_1\dots\nu_k}^{\sigma_1\dots\sigma_k}e_4^\tau[D_\rho, D_\tau]U_{\sigma_1\dots\sigma_k} \\ &= \left[\Pi_\mu^\rho(D_\rho\log\Omega)\Pi_{\nu_1\dots\nu_k}^{\sigma_1\dots\sigma_k}\frac{\mathcal{D}}{\partial\nu}U_{\sigma_1\dots\sigma_k} - \Omega(\Pi_\mu^{\mu'}\mathbf{D}_4\Pi_{\mu'}^\rho)\Pi_{\nu_1\dots\nu_k}^{\sigma_1\dots\sigma_k}D_\rho U_{\sigma_1\dots\sigma_k} \right. \\ &\quad \left. + \Omega\Pi_\mu^\rho\Pi_{\nu_1\dots\nu_k}^{\sigma_1\dots\sigma_k}(D_\rho e_4^\tau)D_\tau U_{\sigma_1\dots\sigma_k} \right] \\ &+ \left[\Pi_\mu^\rho(\Pi_{\nu_1\dots\nu_k}^{\lambda_1\dots\lambda_k}D_\rho\Pi_{\lambda_1\dots\lambda_k}^{\sigma_1\dots\sigma_k})\frac{\mathcal{D}}{\partial\nu}U_{\sigma_1\dots\sigma_k} - \Omega\Pi_\mu^\rho(\Pi_{\nu_1\dots\nu_k}^{\nu'_1\dots\nu'_k}\mathbf{D}_4\Pi_{\nu'_1\dots\nu'_k}^{\sigma_1\dots\sigma_k})D_\rho U_{\sigma_1\dots\sigma_k} \right] \\ &+ \Omega\Pi_\mu^\rho\Pi_{\nu_1\dots\nu_k}^{\sigma_1\dots\sigma_k}e_4^\tau[D_\rho, D_\tau]U_{\sigma_1\dots\sigma_k}\end{aligned}$$

Observe that the first square bracket simplifies. In fact recalling that

$$\Pi_{\mu}^{\mu'} \mathbf{D}_4 \Pi_{\mu'}^{\rho} = \underline{\eta}_{\mu} e_4^{\rho} \quad , \quad \Omega \Pi_{\mu}^{\rho} D_{\rho} e_4^{\tau} = \Omega \chi_{\mu}^{\tau} - \Omega \zeta_{\mu} e_4^{\tau}$$

$$\begin{aligned} \left[(1) \right] &= (\nabla_{\mu} \log \Omega) \Pi_{\nu_1 \dots \nu_k}^{\sigma_1 \dots \sigma_k} \frac{\mathcal{D}}{\partial \nu} U_{\sigma_1 \dots \sigma_k} - \underline{\eta}_{\mu} \Pi_{\nu_1 \dots \nu_k}^{\sigma_1 \dots \sigma_k} \frac{\mathcal{D}}{\partial \nu} U_{\sigma_1 \dots \sigma_k} \\ &+ \Omega \chi_{\mu}^{\tau} \Pi_{\nu_1 \dots \nu_k}^{\sigma_1 \dots \sigma_k} D_{\tau} U_{\sigma_1 \dots \sigma_k} - \zeta_{\mu} \Pi_{\nu_1 \dots \nu_k}^{\sigma_1 \dots \sigma_k} \frac{\mathcal{D}}{\partial \nu} U_{\sigma_1 \dots \sigma_k} \\ &= \Omega \Pi_{\nu_1 \dots \nu_k}^{\sigma_1 \dots \sigma_k} \chi_{\mu}^{\rho} D_{\rho} U_{\sigma_1 \dots \sigma_k} \quad , \end{aligned} \quad (14.16)$$

therefore

$$\begin{aligned} [\nabla_{\mu}, \frac{\mathcal{D}}{\partial \nu}] U_{\nu_1 \dots \nu_k} &= \left[\Pi_{\mu}^{\rho} (\Pi_{\nu_1 \dots \nu_k}^{\lambda_1 \dots \lambda_k} D_{\rho} \Pi_{\lambda_1 \dots \lambda_k}^{\sigma_1 \dots \sigma_k}) \frac{\mathcal{D}}{\partial \nu} U_{\sigma_1 \dots \sigma_k} - \Omega \Pi_{\mu}^{\rho} (\Pi_{\nu_1 \dots \nu_k}^{\nu'_1 \dots \nu'_k} \mathbf{D}_4 \Pi_{\nu'_1 \dots \nu'_k}^{\sigma_1 \dots \sigma_k}) D_{\rho} U_{\sigma_1 \dots \sigma_k} \right. \\ &\quad \left. + \Omega \Pi_{\nu_1 \dots \nu_k}^{\sigma_1 \dots \sigma_k} \chi_{\mu}^{\rho} D_{\rho} U_{\sigma_1 \dots \sigma_k} \right] + \Omega \Pi_{\mu}^{\rho} \Pi_{\nu_1 \dots \nu_k}^{\sigma_1 \dots \sigma_k} e_4^{\tau} [D_{\rho}, D_{\tau}] U_{\sigma_1 \dots \sigma_k} \quad . \end{aligned}$$

Observe now that

$$\Pi_{\mu}^{\rho} (\Pi_{\nu_1 \dots \nu_k}^{\lambda_1 \dots \lambda_k} D_{\rho} \Pi_{\lambda_1 \dots \lambda_k}^{\sigma_1 \dots \sigma_k}) = \sum_{j=1}^k \Pi_{\nu_1 \dots \hat{\nu}_j \dots \nu_k}^{\sigma_1 \dots \hat{\lambda}_j \dots \sigma_k} (\Pi_{\nu_j}^{\lambda_j} \nabla_{\mu} \Pi_{\lambda_j}^{\sigma_j}) = \frac{1}{2} \sum_{j=1}^k \Pi_{\nu_1 \dots \hat{\nu}_j \dots \nu_k}^{\sigma_1 \dots \hat{\lambda}_j \dots \sigma_k} (\chi_{\mu \nu_j} e_4^{\sigma_j} + \chi_{\mu \nu_j} e_3^{\sigma_j})$$

Therefore

$$\begin{aligned} \Pi_{\mu}^{\rho} (\Pi_{\nu_1 \dots \nu_k}^{\lambda_1 \dots \lambda_k} D_{\rho} \Pi_{\lambda_1 \dots \lambda_k}^{\sigma_1 \dots \sigma_k}) \frac{\mathcal{D}}{\partial \nu} U_{\sigma_1 \dots \sigma_k} &= \frac{1}{2} \sum_{j=1}^k \Pi_{\nu_1 \dots \hat{\nu}_j \dots \nu_k}^{\sigma_1 \dots \hat{\lambda}_j \dots \sigma_k} (\chi_{\mu \nu_j} e_4^{\sigma_j} + \chi_{\mu \nu_j} e_3^{\sigma_j}) \frac{\mathcal{D}}{\partial \nu} U_{\sigma_1 \dots \sigma_j \dots \sigma_k} \\ &= \frac{\Omega}{2} \sum_{j=1}^k \Pi_{\nu_1 \dots \hat{\nu}_j \dots \nu_k}^{\sigma_1 \dots \hat{\lambda}_j \dots \sigma_k} \left(\chi_{\mu \nu_j} (D_4 U)_{\sigma_1 \dots \sigma_j \dots \sigma_k} e_4^{\sigma_j} + \chi_{\mu \nu_j} (D_4 U)_{\sigma_1 \dots \sigma_j \dots \sigma_k} e_3^{\sigma_j} \right) \\ &= -\frac{\Omega}{2} \sum_{j=1}^k \Pi_{\nu_1 \dots \hat{\nu}_j \dots \nu_k}^{\sigma_1 \dots \hat{\lambda}_j \dots \sigma_k} \left(\chi_{\mu \nu_j} U_{\sigma_1 \dots \sigma_j \dots \sigma_k} (D_4 e_4)^{\sigma_j} + \chi_{\mu \nu_j} U_{\sigma_1 \dots \sigma_j \dots \sigma_k} (D_4 e_3)^{\sigma_j} \right) \\ &= -\frac{\Omega}{2} \sum_{j=1}^k \Pi_{\nu_1 \dots \hat{\nu}_j \dots \nu_k}^{\sigma_1 \dots \hat{\lambda}_j \dots \sigma_k} \chi_{\mu \nu_j} U_{\sigma_1 \dots \sigma_j \dots \sigma_k} (D_4 e_3)^{\sigma_j} = -\Omega \sum_{j=1}^k \Pi_{\nu_1 \dots \hat{\nu}_j \dots \nu_k}^{\sigma_1 \dots \hat{\lambda}_j \dots \sigma_k} \chi_{\mu \nu_j} \underline{\eta}^{\sigma_j} U_{\sigma_1 \dots \sigma_j \dots \sigma_k} \quad . \end{aligned}$$

Moreover, as

$$\begin{aligned} \Pi_{\nu_1 \dots \nu_k}^{\nu'_1 \dots \nu'_k} \mathbf{D}_4 \Pi_{\nu'_1 \dots \nu'_k}^{\sigma_1 \dots \sigma_k} &= \sum_{j=1}^k \Pi_{\nu_1 \dots \hat{\nu}_j \dots \nu_k}^{\sigma_1 \dots \hat{\sigma}_j \dots \sigma_k} (\Pi_{\nu_j}^{\nu'_j} \mathbf{D}_4 \Pi_{\nu'_j}^{\sigma_j}) \\ &= -\sum_{j=1}^k \Pi_{\nu_1 \dots \hat{\nu}_j \dots \nu_k}^{\sigma_1 \dots \hat{\sigma}_j \dots \sigma_k} \left(\Pi_{\nu_j}^{\nu'_j} (\mathcal{P}_4 \theta^4)_{\nu'_j} e_4^{\sigma_j} + \Pi_{\nu_j}^{\nu'_j} (\mathcal{P}_4 \theta^3)_{\nu'_j} e_3^{\sigma_j} \right) \quad , \end{aligned}$$

$$\begin{aligned}
& \Pi_\mu^\rho (\Pi_{\nu_1 \dots \nu_k}^{\nu'_1 \dots \nu'_k} \mathbf{D}_4 \Pi_{\nu'_1 \dots \nu'_k}^{\sigma_1 \dots \sigma_k}) D_\rho U_{\sigma_1 \dots \sigma_k} = (\Pi_{\nu_1 \dots \nu_k}^{\nu'_1 \dots \nu'_k} \mathbf{D}_4 \Pi_{\nu'_1 \dots \nu'_k}^{\sigma_1 \dots \sigma_k}) (\nabla_\mu U_{\sigma_1 \dots \sigma_k}) \\
& = - \sum_{j=1}^k \Pi_{\nu_1 \dots \hat{\nu}_j \dots \nu_k}^{\sigma_1 \dots \hat{\sigma}_j \dots \sigma_k} \left(\Pi_{\nu_j}^{\nu'_j} (\mathbf{D}_4 \theta^4)_{\nu'_j} e_4^{\sigma_j} + \Pi_{\nu_j}^{\nu'_j} (\mathbf{D}_4 \theta^3)_{\nu'_j} e_3^{\sigma_j} \right) (\nabla_\mu U_{\sigma_1 \dots \sigma_k}) \\
& = - \sum_{j=1}^k \Pi_{\nu_1 \dots \hat{\nu}_j \dots \nu_k}^{\sigma_1 \dots \hat{\sigma}_j \dots \sigma_k} \left(\Pi_{\nu_j}^{\nu'_j} (\mathbf{D}_4 \theta^4)_{\nu'_j} \right) (\nabla_\mu U_{\sigma_1 \dots \sigma_j \dots \sigma_k}) e_4^{\sigma_j} \\
& = \sum_{j=1}^k \Pi_{\nu_1 \dots \hat{\nu}_j \dots \nu_k}^{\sigma_1 \dots \hat{\sigma}_j \dots \sigma_k} \left(\Pi_{\nu_j}^{\nu'_j} (\mathbf{D}_4 \theta^4)_{\nu'_j} \right) U_{\sigma_1 \dots \sigma_j \dots \sigma_k} (\nabla_\mu e_4)^{\sigma_j} \\
& = - \frac{1}{2} \sum_{j=1}^k \Pi_{\nu_1 \dots \hat{\nu}_j \dots \nu_k}^{\sigma_1 \dots \hat{\sigma}_j \dots \sigma_k} \left(\mathbf{g}(\mathbf{D}_4 e_3, e_C) \theta_{\nu_j}^C \right) U_{\sigma_1 \dots \sigma_j \dots \sigma_k} (\nabla_\mu e_4)^{\sigma_j} \\
& = - \sum_{j=1}^k \Pi_{\nu_1 \dots \hat{\nu}_j \dots \nu_k}^{\sigma_1 \dots \hat{\sigma}_j \dots \sigma_k} \underline{\eta}_{\nu_j} U_{\sigma_1 \dots \sigma_j \dots \sigma_k} (\nabla_\mu e_4)^{\sigma_j} = - \sum_{j=1}^k \Pi_{\nu_1 \dots \hat{\nu}_j \dots \nu_k}^{\sigma_1 \dots \hat{\sigma}_j \dots \sigma_k} \chi_\mu^{\sigma_j} \underline{\eta}_{\nu_j} U_{\sigma_1 \dots \sigma_j \dots \sigma_k}.
\end{aligned}$$

Therefore

$$\Omega \Pi_\mu^\rho (\Pi_{\nu_1 \dots \nu_k}^{\nu'_1 \dots \nu'_k} \mathbf{D}_4 \Pi_{\nu'_1 \dots \nu'_k}^{\sigma_1 \dots \sigma_k}) D_\rho U_{\sigma_1 \dots \sigma_k} = -\Omega \sum_{j=1}^k \Pi_{\nu_1 \dots \hat{\nu}_j \dots \nu_k}^{\sigma_1 \dots \hat{\sigma}_j \dots \sigma_k} \chi_\mu^{\sigma_j} \underline{\eta}_{\nu_j} U_{\sigma_1 \dots \sigma_j \dots \sigma_k}$$

and the final result is:¹¹¹

$$\begin{aligned}
[\nabla_\mu, \frac{\mathcal{D}}{\partial \nu}] U_{\nu_1 \dots \nu_k} &= \left[\Pi_\mu^\rho (\Pi_{\nu_1 \dots \nu_k}^{\lambda_1 \dots \lambda_k} D_\rho \Pi_{\lambda_1 \dots \lambda_k}^{\sigma_1 \dots \sigma_k}) \frac{\mathcal{D}}{\partial \nu} U_{\sigma_1 \dots \sigma_k} - \Omega \Pi_\mu^\rho (\Pi_{\nu_1 \dots \nu_k}^{\nu'_1 \dots \nu'_k} \mathbf{D}_4 \Pi_{\nu'_1 \dots \nu'_k}^{\sigma_1 \dots \sigma_k}) D_\rho U_{\sigma_1 \dots \sigma_k} \right. \\
&\quad \left. + \Omega \Pi_{\nu_1 \dots \nu_k}^{\sigma_1 \dots \sigma_k} \chi_\mu^\rho D_\rho U_{\sigma_1 \dots \sigma_k} \right] + \Omega \Pi_\mu^\rho \Pi_{\nu_1 \dots \nu_k}^{\sigma_1 \dots \sigma_k} e_4^\tau [D_\rho, D_\tau] U_{\sigma_1 \dots \sigma_k} \\
&= -\Omega \left[\sum_{j=1}^k \Pi_{\nu_1 \dots \hat{\nu}_j \dots \nu_k}^{\sigma_1 \dots \hat{\sigma}_j \dots \sigma_k} \left(\chi_{\mu \nu_j} \underline{\eta}^{\sigma_j} - \chi_\mu^{\sigma_j} \underline{\eta}_{\nu_j} \right) U_{\sigma_1 \dots \sigma_j \dots \sigma_k} - \Pi_{\nu_1 \dots \nu_k}^{\sigma_1 \dots \sigma_k} \chi_\mu^\rho (\nabla_\rho U)_{\sigma_1 \dots \sigma_k} \right] \\
&\quad - \Omega \Pi_{\nu_1 \dots \nu_k}^{\sigma_1 \dots \sigma_k} \Pi_\mu^\rho e_4^\tau [D_\tau, D_\rho] U_{\sigma_1 \dots \sigma_k} \quad . \tag{14.17}
\end{aligned}$$

The last term can be rewritten in the following way

$$\begin{aligned}
& -\Omega \Pi_{\nu_1 \dots \nu_k}^{\sigma_1 \dots \sigma_k} \Pi_\mu^\rho e_4^\tau [D_\tau, D_\rho] U_{\sigma_1 \dots \sigma_k} = -\Omega \Pi_{\nu_1 \dots \nu_k}^{\sigma_1 \dots \sigma_k} \theta_\mu^C e_4^\tau [D_\tau, D_\rho] U_{\sigma_1 \dots \sigma_k} \\
& = -\Omega \Pi_{\nu_1 \dots \nu_k}^{\sigma_1 \dots \sigma_k} \theta_\mu^C \sum_{i=1}^k R^{\tilde{\sigma}_i}(e_C, e_4)_{\sigma_i} U_{\sigma_1 \dots \tilde{\sigma}_i \dots \sigma_k} = -\Omega \sum_{i=1}^k \Pi_{\nu_1 \dots \hat{\nu}_i \dots \nu_k}^{\sigma_1 \dots \hat{\sigma}_i \dots \sigma_k} \theta_\mu^C \Pi_{\nu_i}^{\sigma_i} R^{\tilde{\sigma}_i}(e_C, e_4)_{\sigma_i} U_{\sigma_1 \dots \tilde{\sigma}_i \dots \sigma_k} \\
& = -\Omega \theta_\mu^C \sum_{i=1}^k \Pi_{\nu_1 \dots \hat{\nu}_i \dots \nu_k}^{\sigma_1 \dots \hat{\sigma}_i \dots \sigma_k} \Pi_{\nu_i}^{\sigma_i} R^{\tilde{\sigma}_i}(e_C, e_4)_{\sigma_i} U_{\sigma_1 \dots \tilde{\sigma}_i \dots \sigma_k} = -\Omega \theta_\mu^C \sum_{i=1}^k \theta_{\nu_i}^D R^{\sigma_i}(\cdot, e_C, e_4, e_D) U_{\nu_1 \dots \sigma_i \dots \nu_k}
\end{aligned}$$

and

$$[\nabla_\mu, \frac{\mathcal{D}}{\partial \nu}] U_{\nu_1 \dots \nu_k} = -\Omega \left[\sum_{j=1}^k \left(\chi_{\mu \nu_j} \underline{\eta}^{\sigma_j} - \chi_\mu^{\sigma_j} \underline{\eta}_{\nu_j} \right) U_{\nu_1 \dots \sigma_j \dots \nu_k} - \chi_\mu^\rho (\nabla_\rho U)_{\nu_1 \dots \nu_k} \right]$$

¹¹¹In our notations $(\nabla_\mu U_{\sigma_1 \dots \sigma_k})$ is not in general S -tangent, while $(\nabla_\mu U)_{\sigma_1 \dots \sigma_k}$ is.

$$\begin{aligned}
& -\Omega \theta_\mu^C \sum_{j=1}^k \theta_{\nu_j}^D R^{\sigma_j}(\cdot, e_C, e_4, e_D) U_{\nu_1 \dots \sigma_j \dots \nu_k} \\
& = -\Omega \sum_{j=1}^k \left[(\chi_{\mu \nu_j} \underline{\eta}^{\sigma_j} - \chi_\mu^{\sigma_j} \underline{\eta}_{\nu_j}) + \theta_\mu^C \theta_{\nu_j}^D R^{\sigma_j}(\cdot, e_C, e_4, e_D) \right] U_{\nu_1 \dots \sigma_j \dots \nu_k} + \Omega \chi_\mu^\rho (\nabla_\rho U)_{\nu_1 \dots \nu_k} \\
& = -\sum_{j=1}^k C_{\mu \nu_j}^{\sigma_j} U_{\nu_1 \dots \sigma_j \dots \nu_k} + \Omega \chi_\mu^\rho (\nabla_\rho U)_{\nu_1 \dots \nu_k} , \tag{14.18}
\end{aligned}$$

$$\text{with} \quad C_{\mu\nu}^\sigma = \Omega \left[(\chi_{\mu\nu} \underline{\eta}^\sigma - \chi_\mu^\sigma \underline{\eta}_\nu) + \theta_\mu^C \theta_\nu^D R^\sigma(\cdot, e_C, e_4, e_D) \right] \tag{14.19}$$

and the proof of 14.1 is achieved.

The proof of equation 14.5 is similar and we do not report it here.

Next we consider the commutator $[\nabla_\alpha, \nabla_\mu]$ applied to the generic tensor $U_{\nu_1 \nu_2 \dots \nu_q}$

$$\begin{aligned}
\nabla_\alpha \nabla_\mu U_{\nu_1 \nu_2 \dots \nu_q} &= \nabla_\alpha \Pi_\mu^\sigma \Pi_{\nu_1 \dots \nu_q}^{\sigma_1 \dots \sigma_q} D_\sigma U_{\sigma_1 \sigma_2 \dots \sigma_q} = \Pi_\alpha^\rho \Pi_{\nu_1 \dots \nu_q}^{\lambda_1 \dots \lambda_q} D_\rho \Pi_\mu^\sigma \Pi_{\lambda_1 \dots \lambda_q}^{\sigma_1 \dots \sigma_q} D_\sigma U_{\sigma_1 \sigma_2 \dots \sigma_q} \\
&= (\Pi_\alpha^\rho \Pi_{\nu_1 \dots \nu_q}^{\lambda_1 \dots \lambda_q} D_\rho \Pi_\mu^\sigma \Pi_{\lambda_1 \dots \lambda_q}^{\sigma_1 \dots \sigma_q}) D_\sigma U_{\sigma_1 \sigma_2 \dots \sigma_q} + \Pi_\alpha^\rho \Pi_{\nu_1 \dots \nu_q}^{\lambda_1 \dots \lambda_q} \Pi_\mu^\sigma \Pi_{\lambda_1 \dots \lambda_q}^{\sigma_1 \dots \sigma_q} D_\rho D_\sigma U_{\sigma_1 \sigma_2 \dots \sigma_q} \\
\nabla_\mu \nabla_\alpha U_{\nu_1 \nu_2 \dots \nu_q} &= \nabla_\mu \Pi_\alpha^\sigma \Pi_{\nu_1 \dots \nu_q}^{\sigma_1 \dots \sigma_q} D_\sigma U_{\sigma_1 \sigma_2 \dots \sigma_q} = \Pi_\mu^\sigma \Pi_{\nu_1 \dots \nu_q}^{\lambda_1 \dots \lambda_q} D_\sigma \Pi_\alpha^\rho \Pi_{\lambda_1 \dots \lambda_q}^{\sigma_1 \dots \sigma_q} D_\rho U_{\sigma_1 \sigma_2 \dots \sigma_q} \\
&= (\Pi_\mu^\sigma \Pi_{\nu_1 \dots \nu_q}^{\lambda_1 \dots \lambda_q} D_\sigma \Pi_\alpha^\rho \Pi_{\lambda_1 \dots \lambda_q}^{\sigma_1 \dots \sigma_q}) D_\rho U_{\sigma_1 \sigma_2 \dots \sigma_q} + \Pi_\mu^\sigma \Pi_{\nu_1 \dots \nu_q}^{\lambda_1 \dots \lambda_q} \Pi_\alpha^\rho \Pi_{\lambda_1 \dots \lambda_q}^{\sigma_1 \dots \sigma_q} D_\sigma D_\rho U_{\sigma_1 \sigma_2 \dots \sigma_q} .
\end{aligned}$$

Therefore

$$\begin{aligned}
& [\nabla_\alpha, \nabla_\mu] U_{\nu_1 \nu_2 \dots \nu_q} = \\
& \left[(\Pi_\alpha^\rho \Pi_{\nu_1 \dots \nu_q}^{\lambda_1 \dots \lambda_q} D_\rho \Pi_\mu^\sigma \Pi_{\lambda_1 \dots \lambda_q}^{\sigma_1 \dots \sigma_q}) - (\Pi_\mu^\sigma \Pi_{\nu_1 \dots \nu_q}^{\lambda_1 \dots \lambda_q} D_\sigma \Pi_\alpha^\rho \Pi_{\lambda_1 \dots \lambda_q}^{\sigma_1 \dots \sigma_q}) \right] D_\sigma U_{\sigma_1 \sigma_2 \dots \sigma_q} \\
& + \Pi_\alpha^\rho \Pi_\mu^\sigma \Pi_{\nu_1 \dots \nu_q}^{\sigma_1 \dots \sigma_q} [D_\rho, D_\sigma] U_{\sigma_1 \sigma_2 \dots \sigma_q} \\
& = [(\Pi_\alpha^\rho D_\rho \Pi_\mu^\sigma) - (\Pi_\mu^\sigma D_\rho \Pi_\alpha^\rho)] \Pi_{\nu_1 \dots \nu_q}^{\sigma_1 \dots \sigma_q} D_\sigma U_{\sigma_1 \sigma_2 \dots \sigma_q} + \Pi_\alpha^\rho \Pi_\mu^\sigma \Pi_{\nu_1 \dots \nu_q}^{\sigma_1 \dots \sigma_q} [D_\rho, D_\sigma] U_{\sigma_1 \sigma_2 \dots \sigma_q} \\
& = [(\Pi_\alpha^\rho D_\rho \Pi_\mu^\sigma) - (\Pi_\mu^\sigma D_\rho \Pi_\alpha^\rho)] \Pi_{\nu_1 \dots \nu_q}^{\sigma_1 \dots \sigma_q} D_\sigma U_{\sigma_1 \sigma_2 \dots \sigma_q} + \Pi_\alpha^\rho \Pi_\mu^\sigma [D_\rho, D_\sigma] \Pi_{\nu_1 \dots \nu_q}^{\sigma_1 \dots \sigma_q} U_{\sigma_1 \sigma_2 \dots \sigma_q} \\
& = [(\Pi_\alpha^\rho D_\rho \Pi_\mu^\sigma) - (\Pi_\mu^\sigma D_\rho \Pi_\alpha^\rho)] \Pi_{\nu_1 \dots \nu_q}^{\sigma_1 \dots \sigma_q} D_\sigma U_{\sigma_1 \sigma_2 \dots \sigma_q} + \Pi_\alpha^\rho \Pi_\mu^\sigma [D_\rho, D_\sigma] U_{\nu_1 \nu_2 \dots \nu_q}
\end{aligned}$$

Computing the first terms we obtain after an easy, but long computation

$$[(\Pi_\alpha^\rho D_\rho \Pi_\mu^\sigma) - (\Pi_\mu^\sigma D_\rho \Pi_\alpha^\rho)] = [(\theta_\mu^3 \underline{\chi}_\alpha^\beta + \theta_\mu^4 \chi_\alpha^\beta) - (\theta_\alpha^3 \underline{\chi}_\mu^\beta + \theta_\alpha^4 \chi_\mu^\beta)] \Pi_\beta^\sigma , \tag{14.20}$$

therefore

$$\begin{aligned}
& [(\Pi_\alpha^\rho D_\rho \Pi_\mu^\sigma) - (\Pi_\mu^\sigma D_\rho \Pi_\alpha^\rho)] \Pi_{\nu_1 \dots \nu_q}^{\sigma_1 \dots \sigma_q} D_\sigma U_{\sigma_1 \sigma_2 \dots \sigma_q} \\
& = [(\theta_\mu^3 \underline{\chi}_\alpha^\beta + \theta_\mu^4 \chi_\alpha^\beta) - (\theta_\alpha^3 \underline{\chi}_\mu^\beta + \theta_\alpha^4 \chi_\mu^\beta)] \Pi_\beta^\sigma \Pi_{\nu_1 \dots \nu_q}^{\sigma_1 \dots \sigma_q} D_\sigma U_{\sigma_1 \sigma_2 \dots \sigma_q} \\
& = [(\theta_\mu^3 \underline{\chi}_\alpha^\beta + \theta_\mu^4 \chi_\alpha^\beta) - (\theta_\alpha^3 \underline{\chi}_\mu^\beta + \theta_\alpha^4 \chi_\mu^\beta)] \nabla_\beta U_{\nu_1 \nu_2 \dots \nu_q} \tag{14.21}
\end{aligned}$$

and the last term of

$$\begin{aligned} \Pi_\alpha^\rho \Pi_\mu^\sigma [D_\rho, D_\sigma] U_{\nu_1 \nu_2 \dots \nu_q} &= \theta_\alpha^C \theta_\mu^D e_C^\rho e_D^\sigma [D_\rho, D_\sigma] U_{\nu_1 \nu_2 \dots \nu_q} \\ &= \theta_\alpha^C \theta_\mu^D \sum_{i=1}^q R^{\tilde{\nu}_i}(\cdot, e_C, e_D, \cdot)_{\nu_i} U_{\nu_1 \nu_2 \dots \tilde{\nu}_i \dots \nu_q} = -\theta_\alpha^C \theta_\mu^D \sum_{i=1}^q R^{\tilde{\nu}_i}(\cdot, e_C, \cdot, e_D)_{\nu_i} U_{\nu_1 \nu_2 \dots \tilde{\nu}_i \dots \nu_q} \end{aligned} \quad (14.22)$$

so that finally

$$\begin{aligned} &[\nabla_\alpha, \nabla_\mu] U_{\nu_1 \nu_2 \dots \nu_q} \\ &= \left[(\theta_\mu^3 \chi_\alpha^\beta + \theta_\mu^4 \chi_\alpha^\beta) - (\theta_\alpha^3 \chi_\mu^\beta + \theta_\alpha^4 \chi_\mu^\beta) \right] \nabla_\beta U_{\nu_1 \nu_2 \dots \nu_q} - \theta_\alpha^C \theta_\mu^D \sum_{i=1}^q R^{\tilde{\nu}_i}(\cdot, e_C, \cdot, e_D)_{\nu_i} U_{\nu_1 \nu_2 \dots \tilde{\nu}_i \dots \nu_q} . \end{aligned} \quad (14.23)$$

Which is formula 14.9.

Some more commutation relations.

We start considering

$$[\frac{\mathcal{D}}{\partial \nu}, \mathcal{L}_O] U$$

where U is a one form, $U = U_\mu dx^\mu$, \mathcal{L}_O a Lie derivative with respect to a $O^{(i)}$, $\mathcal{L}_O \equiv \mathcal{L}_{O^{(i)}}$, and $\frac{\mathcal{D}}{\partial \nu} = \Omega D_4$. We have

$$\begin{aligned} (\mathcal{L}_O U)(e_A) &= \partial_O(U(e_A)) - U([O, e_A]) = D_O U(e_A) + U(D_{e_A} O) \\ &= (\nabla_O U)(e_A) + \mathbf{g}(e_B, \nabla_{e_A} O) U(e_B) . \end{aligned} \quad (14.24)$$

Therefore

$$\mathcal{L}_O U(\cdot) = \nabla_O U(\cdot) + \mathbf{g}(e_B, \nabla \cdot O) U(e_B) = \nabla_O U(\cdot) + (U \cdot \mathcal{H})(\cdot) \quad (14.25)$$

Observe that \mathcal{H}^{ab} has the structure of a connection coefficient, in fact

$$\mathcal{H}_{ab} = \mathbf{g}(\nabla_a O, \frac{\partial}{\partial x^b}) \quad (14.26)$$

$$\begin{aligned} \frac{\mathcal{D}}{\partial \nu} \mathcal{L}_O U &= \frac{\mathcal{D}}{\partial \nu} \nabla_O U + \frac{\mathcal{D}}{\partial \nu} U \cdot \mathcal{H} + U \cdot \frac{\mathcal{D}}{\partial \nu} \mathcal{H} \\ \mathcal{L}_O \frac{\mathcal{D}}{\partial \nu} U &= \nabla_O \frac{\mathcal{D}}{\partial \nu} U + (\frac{\mathcal{D}}{\partial \nu} U \cdot \mathcal{H}) . \end{aligned} \quad (14.27)$$

Therefore

$$[\frac{\mathcal{D}}{\partial \nu}, \mathcal{L}_O] U = [\frac{\mathcal{D}}{\partial \nu}, \nabla_O] U + U \cdot \frac{\mathcal{D}}{\partial \nu} \mathcal{H} = (\frac{\mathcal{D}}{\partial \nu} O) \cdot \nabla U + (O \cdot [\frac{\mathcal{D}}{\partial \nu}, \nabla]) U + U \cdot \frac{\mathcal{D}}{\partial \nu} \mathcal{H}$$

and expressing the components

$$[\frac{\mathcal{D}}{\partial \nu}, \mathcal{L}_O] U_\mu = \left(\frac{\mathcal{D}}{\partial \nu} O \right)^\lambda \nabla_\lambda U_\mu + O^\sigma [\frac{\mathcal{D}}{\partial \nu}, \nabla]_\sigma U_\mu + U_\rho \left(\frac{\mathcal{D}}{\partial \nu} \mathcal{H} \right)_\mu^\rho$$

Moreover

$$O^\sigma [\frac{\mathcal{D}}{\partial \nu}, \nabla]_\sigma U_\mu = (O^\sigma C_{\sigma\mu}^\tau) U_\tau - (O^\sigma \Omega \chi_\sigma^\rho) \nabla_\rho U_\mu , \quad (14.28)$$

$$\begin{aligned} \left(\frac{\mathcal{D}}{\partial \nu} O \right)^\lambda &= \Omega(D_4 O)_\mu \delta^{\mu\lambda} = -\frac{1}{2} \Omega(D_4 O)_\mu (e_3^\mu e_4^\lambda + e_4^\mu e_3^\lambda) + \Omega(D_4 O)_\mu e_A^\mu e_A^\lambda \\ &= -\frac{1}{2} \Omega \mathbf{g}(D_4 O, e_3) e_4^\lambda + \Omega \mathbf{g}(D_4 O, e_A) e_A^\lambda = \Omega \mathbf{g}(\underline{\eta}, O) e_4^\lambda + \Omega \chi_{\mu\lambda} O^\mu e_A^\lambda \end{aligned} \quad (14.29)$$

and

$$\left(\frac{\mathcal{D}}{\partial \nu} O \right)^\lambda \nabla_\lambda U_\mu = (O^\nu \Omega \chi_\nu^\lambda) \nabla_\lambda U_\mu + (g(\underline{\eta}, O) e_4^\lambda) \nabla_\lambda U_\mu = (O^\nu \Omega \chi_\nu^\lambda) \nabla_\lambda U_\mu . \quad (14.30)$$

Therefore

$$\begin{aligned} [\frac{\mathcal{D}}{\partial \nu}, \mathcal{L}_O] U &= \left((O^\sigma C_{\sigma\mu}^\tau) + \frac{\mathcal{D}}{\partial \nu} \mathcal{H}_\mu^\tau \right) U_\tau + (O^\nu \Omega \chi_\nu^\lambda) \nabla_\lambda U_\mu - (O^\sigma \Omega \chi_\sigma^\rho) \nabla_\rho U_\mu \\ &= \left((O^\sigma C_{\sigma\mu}^\tau) + \frac{\mathcal{D}}{\partial \nu} \mathcal{H}_\mu^\tau \right) U_\tau \end{aligned}$$

which we write in a compact way as

$$[\frac{\mathcal{D}}{\partial \nu}, \mathcal{L}_O] U = E_\mu^\tau U_\tau , \quad (14.31)$$

where

$$E_\mu^\tau \equiv \left((O^\sigma C_{\sigma\mu}^\tau) + \frac{\mathcal{D}}{\partial \nu} \mathcal{H}_\mu^\tau \right) .$$

With similar calculations we obtain

$$[\frac{\mathcal{D}}{\partial \lambda}, \mathcal{L}_O] U = \underline{E}_\mu^\tau U_\tau , \quad (14.32)$$

where

$$\underline{E}_\mu^\tau \equiv \left((O^\sigma \underline{C}_{\sigma\mu}^\tau) + \frac{\mathcal{D}}{\partial \nu} \mathcal{H}_\mu^\tau \right) .$$

Finally we consider $[\nabla, \mathcal{L}_O] U_{\alpha\beta}$ and $[\mathfrak{d}\mathfrak{v}, \mathcal{L}_O] U_{\alpha\beta}$ with $U_{\alpha\beta}$ a symmetric 2-form.

$$\begin{aligned} ([\nabla_\nu, \mathcal{L}_O] U)_{ab} &= (\nabla_\nu O)^\lambda \nabla_\lambda U_{ab} + O^\sigma [\nabla_\nu, \nabla_\sigma] U_{ab} + 2U_{a\rho} (\nabla_\nu \mathcal{H}_b^\rho) = \\ &= O^\rho \hat{C}_{\rho\nu a}^\sigma U_{\sigma b} + 2U_{a\rho} (\nabla_\nu \mathcal{H}_b^\rho) + \sum_{A=1,2} (O^\rho \mathcal{H}_\rho^\beta \theta_{\nu A} \nabla_\beta U_{ab}) \end{aligned} \quad (14.33)$$

which we rewrite symbolically as:

$$([\nabla_\nu, \mathcal{L}_O]U)_{ab} = \hat{E}_{\nu a}^\sigma U_{\sigma b} + \sum_{A=12} O^\rho \hat{\mathcal{H}}_\rho^\beta \theta_{\nu A} \nabla_\beta U_{ab} \quad (14.34)$$

and from it,

$$([\text{div}, \mathcal{L}_O]U)_c = [\nabla_\nu, \mathcal{L}_O]U_c^\nu = O^\rho \hat{C}_{\rho\nu}^\sigma {}^\nu U_{\sigma c} + 2U_{c\rho} (\nabla_\nu \mathcal{H}^{\rho\nu}) + \sum_{A=1,2} (O^\rho \mathcal{H}_\rho^\beta \theta_{\nu A} \nabla_\beta U_c^\nu) \quad (14.35)$$

which we rewrite symbolically as:

$$([\text{div}, \mathcal{L}_O]U)_c = \overline{E}_\nu^\sigma {}^\nu U_{\sigma c} + \sum_{A=12} {}^A \overline{\mathcal{H}}^{\beta\nu} \nabla_\beta U_{\nu c} \quad (14.36)$$

Performing analogous steps we obtain for U_α 1-form

$$\begin{aligned} ([\nabla_\nu, \mathcal{L}_O]U)_c &= (\nabla_\nu O)^\lambda \nabla_\lambda U_c + O^\sigma [\nabla_\nu, \nabla_\sigma]U_c + U_\rho (\nabla_\nu \mathcal{H}_c^\rho) = \\ &= O^\rho \hat{C}_{\rho\nu c}^\sigma U_\sigma + U_\rho (\nabla_\nu \mathcal{H}_c^\rho) + \sum_{A=1,2} (O^\rho \mathcal{H}_\rho^\beta \theta_{\nu A} \nabla_\beta U_c) \end{aligned} \quad (14.37)$$

which we rewrite symbolically as:

$$[\nabla_\nu, \mathcal{L}_O]U_c = \hat{E}_{\nu c}^\sigma U_\sigma + \sum_{A=12} O^\rho \hat{\mathcal{H}}_\rho^\beta \theta_{\nu A} \nabla_\beta U_c \quad (14.38)$$

and from it,

$$([\text{div}, \mathcal{L}_O]U) = [\nabla_\nu, \mathcal{L}_O]U^\nu = O^\rho \hat{C}_{\rho\nu}^\sigma {}^\nu U_\sigma + U_\rho (\nabla_\nu \mathcal{H}^{\rho\nu}) + \sum_{A=1,2} (O^\rho \mathcal{H}_\rho^\beta \theta_{\nu A} \nabla_\beta U^\nu) \quad (14.39)$$

which we rewrite symbolically as:

$$([\text{div}, \mathcal{L}_O]U) = \tilde{E}_\nu^\sigma {}^\nu U_\sigma + \sum_{A=12} {}^A \tilde{\mathcal{H}}^{\beta\nu} \nabla_\beta U_\nu \quad (14.40)$$

Moreover we need an expression for $[\text{curl}, \mathcal{L}_O]U$, with U 1-form, to do this we exploit the expression 14.41

$$\begin{aligned}
([\text{curl}, \mathcal{L}_O]U) &= \epsilon^{\nu c} (\nabla_\nu O)^\lambda \nabla_\lambda U_c + \epsilon^{\nu c} O^\sigma [\nabla_\nu, \nabla_\sigma] U_c + \epsilon^{\nu c} U_\rho (\nabla_\nu \mathcal{H}_c^\rho) = \\
&= \epsilon^{\nu c} O^\rho \hat{C}_{\rho\nu c}^\sigma U_\sigma + \epsilon^{\nu c} U_\rho (\nabla_\nu \mathcal{H}_c^\rho) + \epsilon^{\nu c} \sum_{A=1,2} (O^\rho \mathcal{H}_\rho^\beta \theta_{\nu A} \nabla_\beta U_c) \quad (14.41)
\end{aligned}$$

As before we write the last expression in a symbolical way

$$([\text{curl}, \mathcal{L}_O]U) = \hat{E}_\nu^\sigma \nabla_\sigma U_c + \sum_{A=1,2} {}^A \hat{\mathcal{H}}^{\beta c} \nabla_\beta U_c \quad (14.42)$$

Remarks:

i) It is important to remark that the difference of the commutator $[\mathcal{L}_O^{N-1}, \frac{\mathcal{D}}{\partial\nu}]$ with respect to $[\nabla^{N-1}, \frac{\mathcal{D}}{\partial\nu}]$ is the reason of the missing of the factor $(N-1)$ in front of $\text{tr}\chi$ in the transport equation of $\mathcal{L}_O^{N-1}\psi$, as expected and connected to the absence of extra r -decays for $\hat{\mathcal{L}}_O^{N-1}\psi$.

ii) We expect, as we show next, that we could perform all the estimates for the \mathcal{L}_O^N of the connection coefficients in the same way obtaining for their $|\cdot|_{p,S}$ norms the estimates assumed before for the ∇^N derivatives, with a different constant in front. In fact, also in this case, the commutators have the same structure of the commutators with the ∇ derivatives, namely a combination of products of connection coefficients and metric components time ∇U and a combination of null Riemann components times U .

14.1 Proof of Lemma 9.2:

Let us look at the transport equation of $\mathcal{L}_O^{N-1}\nabla\text{tr}\chi$ or more carefully to the one of $\mathcal{L}_O^{N-1}\psi$, the equation for ψ is

$$\frac{\mathcal{D}}{\partial\nu}\psi + \frac{3}{2}\Omega\text{tr}\chi\psi = -\nabla|\hat{\chi}|^2 - \eta|\hat{\chi}|^2 - \Omega\hat{\chi} \cdot \psi + \text{tr}\chi(\hat{\chi} \cdot \underline{\eta}) - \text{tr}\chi\beta \quad (14.43)$$

Applying \mathcal{L}_O^{N-1} to equation 14.43 we obtain:

$$\mathcal{L}_O^{N-1}\frac{\mathcal{D}\psi}{\partial\nu} = \frac{\mathcal{D}}{\partial\nu}(\mathcal{L}_O^{N-1}\psi) + [\mathcal{L}_O^{N-1}, \frac{\mathcal{D}}{\partial\nu}]\psi = \frac{\mathcal{D}}{\partial\nu}(\mathcal{L}_O^{N-1}\psi) + \sum_{k=0}^{N-2} \mathcal{L}_O^k [\mathcal{L}_O, \frac{\mathcal{D}}{\partial\nu}](\mathcal{L}_O^{N-2-k}\psi) \quad .$$

Therefore

$$\begin{aligned}
\frac{\mathcal{D}}{\partial\nu}(\mathcal{L}_O^{N-1}\psi) &= \mathcal{L}_O^{N-1}\frac{\mathcal{D}\psi}{\partial\nu} - \sum_{k=0}^{N-2} \mathcal{L}_O^k [\frac{\mathcal{D}}{\partial\nu}, \mathcal{L}_O](\mathcal{L}_O^{N-2-k}\psi) \\
&= \mathcal{L}_O^{N-1}\frac{\mathcal{D}\psi}{\partial\nu} - \sum_{k=0}^{N-2} \mathcal{L}_O^k E(\mathcal{L}_O^{N-2-k}\psi) \quad . \quad (14.44)
\end{aligned}$$

Observe that

$$\sum_{k=0}^{N-2} \sum_{s=0}^k \binom{k}{s} = \sum_{s=0}^{N-2} \sum_{k=s}^{N-2} \binom{k}{s} = \sum_{s=0}^{N-2} \binom{N-1}{N-2-s}, \quad (14.45)$$

therefore

$$\begin{aligned} \sum_{k=0}^{N-2} \mathcal{L}_O^k E(\mathcal{L}_O^{N-2-k} \psi) &= \sum_{k=0}^{N-2} \sum_{s=0}^k \binom{k}{s} (\mathcal{L}_O^s E)(\mathcal{L}_O^{N-2-s} \psi) \\ &= \sum_{s=0}^{N-2} \sum_{k=s}^{N-2} \binom{k}{s} (\mathcal{L}_O^s E)(\mathcal{L}_O^{N-2-s} \psi) = \sum_{s=0}^{N-2} \binom{N-1}{N-2-s} (\mathcal{L}_O^s E)(\mathcal{L}_O^{N-2-s} \psi) \end{aligned} \quad (14.46)$$

and

$$\begin{aligned} \frac{\mathcal{D}}{\partial \nu}(\mathcal{L}_O^{N-1} \psi) &= \mathcal{L}_O^{N-1} \frac{\mathcal{D} \psi}{\partial \nu} - \sum_{k=0}^{N-2} \mathcal{L}_O^k \left[\frac{\mathcal{D}}{\partial \nu}, \mathcal{L}_O \right] (\mathcal{L}_O^{N-2-k} \psi) \\ &= \mathcal{L}_O^{N-1} \frac{\mathcal{D} \psi}{\partial \nu} - \sum_{k=0}^{N-2} \mathcal{L}_O^k E(\mathcal{L}_O^{N-2-k} \psi) = \mathcal{L}_O^{N-1} \frac{\mathcal{D} \psi}{\partial \nu} - \sum_{s=0}^{N-2} \binom{N-1}{N-2-s} (\mathcal{L}_O^s E)(\mathcal{L}_O^{N-2-s} \psi) \\ &= \mathcal{L}_O^{N-1} \frac{\mathcal{D} \psi}{\partial \nu} - \sum_{q=0}^{N-2} \binom{N-1}{q} (\mathcal{L}_O^{N-2-q} E)(\mathcal{L}_O^q \psi). \end{aligned} \quad (14.47)$$

Therefore

$$\begin{aligned} \frac{\mathcal{D}}{\partial \nu}(\mathcal{L}_O^{N-1} \psi) &= \mathcal{L}_O^{N-1} \left[-\frac{3}{2} \Omega \text{tr} \chi \psi - \nabla |\hat{\chi}|^2 - \eta |\hat{\chi}|^2 - \Omega \hat{\chi} \cdot \psi + \text{tr} \chi (\hat{\chi} \cdot \underline{\eta}) - \text{tr} \chi \beta \right] \\ &\quad - \sum_{q=0}^{N-2} \binom{N-1}{q} (\mathcal{L}_O^{N-2-q} E)(\mathcal{L}_O^q \psi). \end{aligned} \quad (14.48)$$

which we rewrite as

$$\begin{aligned} \frac{\mathcal{D}}{\partial \nu}(\mathcal{L}_O^{N-1} \psi) &+ \frac{3}{2} \Omega \text{tr} \chi (\mathcal{L}_O^{N-1} \psi) = -\Omega \hat{\chi} \cdot (\mathcal{L}_O^{N-1} \psi) - \mathcal{L}_O^{N-1} \nabla |\hat{\chi}|^2 \\ &- \frac{3}{2} \sum_{q=1}^{N-1} \binom{N-1}{q} (\mathcal{L}_O^q \Omega \text{tr} \chi) (\mathcal{L}_O^{N-1-q} \psi) - \mathcal{L}_O^{N-1} [-\eta |\hat{\chi}|^2 - \Omega \hat{\chi} \cdot \psi + \text{tr} \chi (\hat{\chi} \cdot \underline{\eta}) - \text{tr} \chi \beta] \\ &- \sum_{q=0}^{N-2} \binom{N-1}{q} (\mathcal{L}_O^{N-2-q} E)(\mathcal{L}_O^q \psi) = \\ &= -\Omega \hat{\chi} \cdot (\mathcal{L}_O^{N-1} \psi) - 2 \hat{\chi} \mathcal{L}_O^{N-1} (\nabla \hat{\chi}) - \text{tr} \chi \mathcal{L}_O^{N-1} \beta + 2 \sum_{k=1}^{N-1} \binom{N-1}{k} \mathcal{L}_O^{N-1-k} (\nabla \hat{\chi}) \mathcal{L}_O^k \hat{\chi} \\ &- \frac{3}{2} \sum_{q=1}^{N-1} \binom{N-1}{q} (\mathcal{L}_O^q \Omega \text{tr} \chi) (\mathcal{L}_O^{N-1-q} \psi) + \sum_{k=0}^{N-1} \binom{N-1}{k} \mathcal{L}_O^{N-1-k} (|\hat{\chi}|^2) \mathcal{L}_O^k \eta \end{aligned}$$

$$\begin{aligned}
& + \sum_{q=1}^{N-1} \binom{N-1}{q} (\mathcal{L}_O^q(\Omega\hat{\chi})(\mathcal{L}_O^{N-1-q}\psi) - \sum_{q=0}^{N-2} \binom{N-1}{q} (\mathcal{L}_O^{N-2-q}E)(\mathcal{L}_O^q\psi) \\
& - \sum_{q=0}^{N-1} \binom{N-1}{q} (\mathcal{L}_O^q(\text{tr}\chi)(\mathcal{L}_O^{N-1-q}\hat{\chi}\eta) + \sum_{q=1}^{N-1} \binom{N-1}{q} (\mathcal{L}_O^{N-2-q}\beta)(\mathcal{L}_O^q\text{tr}\chi) \quad (14.49)
\end{aligned}$$

We rewrite the last formula in a more convenient way:

$$\begin{aligned}
\frac{\mathcal{D}}{\partial\nu}(\mathcal{L}_O^{N-1}\psi) + \frac{3}{2}\overline{\Omega\text{tr}\chi}(\mathcal{L}_O^{N-1}\psi) &= -\Omega\hat{\chi} \cdot (\mathcal{L}_O^{N-1}\psi) - 2\hat{\chi}\mathcal{L}_O^{N-1}(\nabla\hat{\chi}) - \text{tr}\chi\mathcal{L}_O^{N-1}\beta \\
&+ \frac{3}{2}(\overline{\Omega\text{tr}\chi} - \Omega\text{tr}\chi)(\mathcal{L}_O^{N-1}\psi) + \{(good)_1\} \quad (14.50)
\end{aligned}$$

$$\begin{aligned}
\{(good)_1\} &= 2 \sum_{k=1}^{N-1} \binom{N-1}{k} \mathcal{L}_O^{N-1-k}(\nabla\hat{\chi})\mathcal{L}_O^k\hat{\chi} \\
&- \frac{3}{2} \sum_{q=1}^{N-1} \binom{N-1}{q} (\mathcal{L}_O^q\Omega^2U)(\mathcal{L}_O^{N-1-q}\psi) + \sum_{k=0}^{N-1} \binom{N-1}{k} \mathcal{L}_O^{N-1-k}(|\hat{\chi}|^2)\mathcal{L}_O^k\eta \\
&+ \sum_{q=1}^{N-1} \binom{N-1}{q} (\mathcal{L}_O^q(\Omega\hat{\chi})(\mathcal{L}_O^{N-1-q}\psi) + \sum_{q=0}^{N-2} \binom{N-1}{q} (\mathcal{L}_O^{N-2-q}E)(\mathcal{L}_O^q\psi) \\
&- \sum_{q=0}^{N-1} \binom{N-1}{q} (\mathcal{L}_O^q(\Omega U)(\mathcal{L}_O^{N-1-q}\hat{\chi}\eta) + \sum_{q=1}^{N-1} \binom{N-1}{q} (\mathcal{L}_O^{N-2-q}\beta)(\mathcal{L}_O^q\text{tr}\chi) \quad (14.51)
\end{aligned}$$

and equation 9.25 of lemma 9.2 is proved. In order to prove inequality 9.26 we have to consider the $|\cdot|_{p,S}$ norms, equation 9.25 implies the following one:

$$\begin{aligned}
\frac{\partial|(\mathcal{L}_O^{N-1}\psi)|^p}{\partial\nu} + p\frac{3}{2}\overline{\Omega\text{tr}\chi}|(\mathcal{L}_O^{N-1}\psi)|^p &\geq -p|\mathcal{L}_O^{N-1}\psi|^p\frac{3}{2}|(\Omega\text{tr}\chi) - (\overline{\Omega\text{tr}\chi})| \\
-p|\mathcal{L}_O^{N-1}\psi|^{p-1}[\Omega\hat{\chi} \cdot (\mathcal{L}_O^{N-1}\psi) &+ 2|\hat{\chi}\mathcal{L}_O^{N-1}(\nabla\hat{\chi})| + |\text{tr}\chi||\mathcal{L}_O^{N-1}\beta| + |\{(good)_1\}|] \quad (14.52)
\end{aligned}$$

which we summarize as

$$\frac{\partial|(\mathcal{L}_O^{N-1}\psi)|^p}{\partial\nu} + p\frac{3}{2}\overline{\Omega\text{tr}\chi}|(\mathcal{L}_O^{N-1}\psi)|^p \geq -p\frac{3}{2}|(\Omega\text{tr}\chi) - (\overline{\Omega\text{tr}\chi})||\mathcal{L}_O^{N-1}\psi|^p - p|\mathcal{L}_O^{N-1}\psi|^{p-1}|F|, \quad (14.53)$$

with

$$|F| = [\Omega\hat{\chi} \cdot (\mathcal{L}_O^{N-1}\psi) + 2|\hat{\chi}\mathcal{L}_O^{N-1}(\nabla\hat{\chi})| + |\text{tr}\chi||\mathcal{L}_O^{N-1}\beta| + |\{(good)_1\}|] . \quad (14.54)$$

From 14.53 and by lemma 4.1.5 [Kl-Ni2]

$$\begin{aligned} \frac{\partial |r^{(3-\frac{2}{p})} \mathcal{L}_O^{N-1} \psi|_{p,S}^p}{\partial \nu} &= \int_S \left[r^{(3-\frac{2}{p})p} \left(\frac{\partial |\mathcal{L}_O^{N-1} \psi|^p}{\partial \nu} + p \frac{3\overline{\Omega \text{tr} \chi}}{2} |\mathcal{L}_O^{N-1} \psi|^p \right) \right] d\mu_\gamma \\ &\geq -p \int_S \left[r^{(3-\frac{2}{p})p} |\mathcal{L}_O^{N-1} \psi|^{p-1} |F| + \left(\frac{3-\frac{2}{p}}{2} |\Omega \text{tr} \chi - \overline{\Omega \text{tr} \chi}| \right) |r^{(3-\frac{2}{p})} \mathcal{L}_O^{N-1} \psi|^p \right] , \end{aligned}$$

which we rewrite

$$- \frac{\partial |r^{(3-\frac{2}{p})} \mathcal{L}_O^{N-1} \psi|_{p,S}^p}{\partial \nu} \leq p \int_S r^{(3-\frac{2}{p})p} |\mathcal{L}_O^{N-1} \psi|^{p-1} |\mathcal{L}| d\mu_\gamma , \quad (14.55)$$

where

$$\begin{aligned} |\mathcal{L}| &= [|\Omega \hat{\chi} \cdot (\mathcal{L}_O^{N-1} \psi)| + |2\hat{\chi} \cdot (\mathcal{L}_O^{N-1} \nabla \hat{\chi})| + |\text{tr} \chi| |\mathcal{L}_O^{N-1} \beta| + |\{(good)_1\}|] \\ &+ \left[\frac{(3-\frac{2}{p})}{2} |\Omega \text{tr} \chi - \overline{\Omega \text{tr} \chi}| |\mathcal{L}_O^{N-1} \psi| \right] . \end{aligned} \quad (14.56)$$

Applying the Holder inequality to the right hand side we obtain

$$\int_S r^{(3-\frac{2}{p})p} |\mathcal{L}_O^{N-1} \psi|^{p-1} |\mathcal{L}| d\mu_\gamma \leq |r^{(3-\frac{2}{p})} \mathcal{L}|_{p,S} |r^{(3-\frac{2}{p})} \mathcal{L}_O^{N-1} \psi|_{p,S}^{p-1} \quad (14.57)$$

and from it inequality 9.27,

$$- \frac{\partial}{\partial \nu} |r^{(3-\frac{2}{p})} \mathcal{L}_O^{N-1} \psi|_{p,S} \leq |r^{(3-\frac{2}{p})} \mathcal{L}|_{p,S} . \quad (14.58)$$

and Lemma 9.2 is proved .

14.2 Proof of Lemma 9.3:

We are left with the problem of estimating the term $|\mathcal{L}_O^{N-1} \nabla |\hat{\chi}|^2|_{p,S}$. We have

$$\mathcal{L}_O^{N-1} \hat{\chi} \cdot \nabla \hat{\chi} = \sum_{q=0}^{N-1} (\mathcal{L}_O^{N-1-q} \hat{\chi}) \cdot (\mathcal{L}_O^q \nabla \hat{\chi}) = \hat{\chi} \cdot (\mathcal{L}_O^{N-1} \nabla \hat{\chi}) + \sum_{q=0}^{N-2} (\mathcal{L}_O^{N-1-q} \hat{\chi}) \cdot (\hat{\mathcal{L}}_O^q \nabla \hat{\chi}) .$$

which implies that we need estimates for the norms of $\mathcal{L}_O^q \nabla \hat{\chi}$. We have them inductively when $q < N-1$, while for $q = N-1$ we have to proceed differently using the Hodge system. Observe that we can write

$$\mathcal{L}_O^{N-1} \nabla \hat{\chi} = \nabla (\mathcal{L}_O^{N-1} \hat{\chi}) + [\mathcal{L}_O^{N-1}, \nabla] \hat{\chi} \quad (14.59)$$

The first term of the r.h.s. can be estimated using the Hodge system for $(\mathcal{L}_O^{N-1} \hat{\chi})$ derived from

$$\mathfrak{d}^* \hat{\chi} = \frac{1}{2} \nabla \text{tr} \chi - \beta - \zeta \cdot \hat{\chi} + \frac{1}{2} \zeta \text{tr} \chi = \frac{\Omega}{2} \psi - \beta - \zeta \cdot \hat{\chi} , \quad (14.60)$$

while the second term written in the following way,

$$[\mathcal{L}_O^{N-1}, \nabla] \hat{\chi} = \sum_{h=0}^{N-2} \mathcal{L}_O^h [\mathcal{L}_O, \nabla] (\hat{\mathcal{L}}_O^{N-2-h} \hat{\chi}) . \quad (14.61)$$

can be treated as done for the term $[\frac{\mathcal{D}}{dv}, \mathcal{L}_O^{N-1}] \psi$, see 14.47, and it involves lower order terms which can be estimated inductively. Let us regroup all the terms involving the commutator $[\nabla, \mathcal{L}_O^{N-1}] \hat{\chi}_{ab}$ and denoting it $\{(good)_2\}$.

We are left with estimating $\nabla(\mathcal{L}_O^{N-1} \hat{\chi})$ and to do this, we have to estimate $\text{div} \mathcal{L}_O^{N-1} \hat{\chi}$. From equation 14.60 it follows,

$$\begin{aligned} \text{div} \mathcal{L}_O^{N-1} \hat{\chi} &= \mathcal{L}_O^{N-1} \text{div} \hat{\chi} + [\text{div}, \mathcal{L}_O^{N-1}] \hat{\chi} = \mathcal{L}_O^{N-1} \left\{ \frac{\Omega}{2} \psi - \beta - \zeta \cdot \hat{\chi} \right\} + [\text{div}, \mathcal{L}_O^{N-1}] \hat{\chi} \\ &= \mathcal{L}_O^{N-1} \frac{\Omega}{2} \psi - \mathcal{L}_O^{N-1} \beta + \{(good)_3\} , \end{aligned} \quad (14.62)$$

with

$$\begin{aligned} \{(good)_3\} &= -\mathcal{L}_O^{N-1} (\zeta \cdot \hat{\chi}) + [\text{div}, \mathcal{L}_O^{N-1}] \hat{\chi} \\ &= \mathcal{L}_O^{N-1} (\zeta \cdot \hat{\chi}) + \sum_{q=0}^{N-2} \left(\binom{N-1}{q} (\mathcal{L}_O^{N-2-q} \overline{E}) (\mathcal{L}_O^q \psi) + \binom{N-1}{q} (\mathcal{L}_O^{N-2-q} \overline{H}) (\mathcal{L}_O^q \nabla \psi) \right) \end{aligned} \quad (14.63)$$

By proposition 4.1.3 of [Kl-Ni2] we have that

$$\begin{aligned} |\nabla \mathcal{L}_O^{N-1} \hat{\chi}|_{p,S} &\leq c \left(|\Omega|_{\infty,S} |\mathcal{L}_O^{N-1} \psi|_{p,S} + |\mathcal{L}_O^{N-1} \beta|_{p,S} + \sum_{k=1}^{N-1} \binom{N-1}{k} |\mathcal{L}_O^k \Omega|_{\infty} |\mathcal{L}_O^{N-1-k} \psi|_{p,S} + |\{(good)_3\}|_{p,S} \right) \\ &= c (|\Omega|_{\infty,S} |\mathcal{L}_O^{N-1} \psi|_{p,S} + |\mathcal{L}_O^{N-1} \beta|_{p,S} + |\{(good)_4\}|_{p,S}) \end{aligned} \quad (14.64)$$

with c a generic constant and

$$|\{(good)_4\}|_{p,S} = \left(\sum_{k=1}^{N-1} \binom{N-1}{k} |\mathcal{L}_O^k \Omega|_{\infty} |\mathcal{L}_O^{N-1-k} \hat{\chi}|_{p,S} + |\{(good)_3\}|_{p,S} \right) \quad (14.65)$$

Collecting equations 14.59 and 14.64 we obtain

$$|\mathcal{L}_O^{N-1} \nabla \hat{\chi}|_{p,S} \leq c (|\Omega|_{\infty,S} |\mathcal{L}_O^{N-1} \psi|_{p,S} + |\mathcal{L}_O^{N-1} \beta|_{p,S} + |\{(good)_2\}|_{p,S} + |\{(good)_4\}|_{p,S}) \quad (14.66)$$

and Lemma 9.3 is proved.

14.3 Proof of Theorem 9.1

In order to prove theorem 9.1 we have to estimate inductively $|\mathcal{L}|_{p,S}$, applying the result of lemma 9.3 to the expression of $|\mathcal{L}|$, see 14.56

$$\begin{aligned}
|\mathcal{L}|_{p,S} &\leq \left(|\Omega \hat{\chi} \cdot (\mathcal{L}_O^{N-1} \psi)|_{p,S} + |2\hat{\chi}|_{\infty,S} |(\mathcal{L}_O^{N-1} \nabla \hat{\chi})|_{p,S} + |\text{tr} \chi|_{\infty,S} |\mathcal{L}_O^{N-1} \beta|_{p,S} + |\{(good)_1\}|_{p,S} \right) \\
&\quad + \left(\frac{(3 - \frac{2}{p})}{2} |\Omega \text{tr} \chi - \overline{\Omega \text{tr} \chi}|_{\infty,S} |\mathcal{L}_O^{N-1} \psi|_{p,S} \right) . \\
&\leq \left(|\Omega \hat{\chi} \cdot (\mathcal{L}_O^{N-1} \psi)|_{p,S} + |\text{tr} \chi|_{\infty,S} |\mathcal{L}_O^{N-1} \beta|_{p,S} + |\{(good)_1\}|_{p,S} \right) \\
&\quad + |2\hat{\chi}|_{\infty,S} (|\Omega|_{\infty,S} |\mathcal{L}_O^{N-1} \psi|_{p,S} + |\mathcal{L}_O^{N-1} \beta|_{p,S} + |\{(good)_4\}|_{p,S}) + \left(\frac{(3 - \frac{2}{p})}{2} |\Omega \text{tr} \chi - \overline{\Omega \text{tr} \chi}|_{\infty,S} |\mathcal{L}_O^{N-1} \psi|_{p,S} \right) = . \\
&\leq c \left(\left(|\hat{\chi}|_{\infty,S} |\Omega|_{\infty,S} + \frac{(3 - \frac{2}{p})}{2} |\Omega \text{tr} \chi - \overline{\Omega \text{tr} \chi}|_{\infty,S} \right) |(\mathcal{L}_O^{N-1} \psi)|_{p,S} + (2|\hat{\chi}|_{\infty,S} + |\text{tr} \chi|_{\infty,S}) |\mathcal{L}_O^{N-1} \beta|_{p,S} \right. \\
&\quad \left. + |\{(good)_1\}|_{p,S} + |\{(good)_4\}|_{p,S} \right)
\end{aligned}$$

which we rewrite, defining,

$$|\mathcal{H}|_{\infty,S} \equiv |\Omega|_{\infty,S} |\hat{\chi}|_{\infty,S} + \frac{(3 - \frac{2}{p})}{2} |\Omega \text{tr} \chi - \overline{\Omega \text{tr} \chi}|_{\infty,S} \quad (14.67)$$

$$|r^{3-\frac{2}{p}} \{(good)_{1+4}\}|_{p,S} \equiv |r^{3-\frac{2}{p}} \{(good)_1\}|_{p,S} + |r^{3-\frac{2}{p}} \{(good)_4\}|_{p,S} ,$$

We have

$$\begin{aligned}
|r^{(3-\frac{2}{p})} \mathcal{L}|_{p,S} &\leq c |\mathcal{H}|_{\infty,S} (\lambda, \nu) |r^{3-\frac{2}{p}} (\mathcal{L}_O^{N-1} \psi)|_{p,S} \\
&\quad + c (2|\hat{\chi}|_{\infty,S} + |\text{tr} \chi|_{\infty,S}) |r^{3-\frac{2}{p}} (\mathcal{L}_O^{N-1} \beta)|_{p,S} + c |r^{3-\frac{2}{p}} \{(good)_{1+4}\}|_{p,S} .
\end{aligned}$$

Hence inequality 14.58 can be written as,

$$\begin{aligned}
& - \frac{\partial}{\partial \nu} |r^{(3-\frac{2}{p})} \mathcal{L}_O^{N-1} \psi|_{p,S} - c |\mathcal{H}|_{\infty,S} |r^{(3-\frac{2}{p})} (\mathcal{L}_O^{N-1} \psi)|_{p,S} \\
& \leq \left[(2|\hat{\chi}|_{\infty,S} + |\text{tr} \chi|_{\infty,S}) |r^{3-\frac{2}{p}} (\mathcal{L}_O^{N-1} \beta)|_{p,S} + c_0 |r^{3-\frac{2}{p}} \{(good)_{1+4}\}|_{p,S} \right] ,
\end{aligned} \quad (14.68)$$

Equation 14.68 is the one we use to prove the estimate for $\mathcal{L}_O^{N-1} \psi$.

We are left with the problem of estimating $|(\mathcal{L}_O^{N-1} \beta)|_{p,S}$. We postpone its estimate and prove the result for $|r^{(N+2-\frac{2}{p})} \mathcal{L}_O^{N-1} \psi|_{p,S}(\lambda, \nu)$ using as inductive assumptions for the connection coefficients with $J < N$, the results of Theorem 3.6.¹¹²

¹¹²These inductive assumptions imply the inductive assumption for ψ ,

$$|r^{3-\frac{2}{p}} \mathcal{L}_O^{J-1} \psi|_{p,S} \leq C_0 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_{0,1}^J} .$$

Integrating inequality 14.68 from ν to ν_* we obtain, applying the Gronwall lemma:

$$|r^{(3-\frac{2}{p})} \mathcal{L}_O^{N-1} \psi|_{p,S}(\lambda, \nu) \leq \left(\exp c \int_{\nu}^{\nu_*} |\mathcal{H}|_{\infty,S}(\nu') d\nu' \right) |r^{(3-\frac{2}{p})} \mathcal{L}_O^{N-1} \psi|_{p,S}(\lambda, \nu_*) \quad (14.69)$$

$$+ C \left(e^{\int_{\nu}^{\nu_*} |\mathcal{H}|_{\infty,S}(\nu') d\nu'} \cdot \left[\int_{\nu}^{\nu_*} d\nu' \left(e^{-\int_{\nu'}^{\nu_*} |\mathcal{H}|_{\infty,S}(\nu'') d\nu''} \right) \cdot (2|\hat{\chi}|_{\infty,S} + |\text{tr}\chi|_{\infty,S}) \right] |r^{3-\frac{2}{p}} (\mathcal{L}_O^{N-1} \beta)|_{p,S} \right. \\ \left. + \int_{\nu}^{\nu_*} d\nu' \left(e^{-\int_{\nu'}^{\nu_*} |\mathcal{H}|_{\infty,S}(\nu'') d\nu''} \right) |r^{3-\frac{2}{p}} \{(good)_{1+4}\}|_{p,S} \right] \\ \leq \left(\exp c \int_{\nu}^{\nu_*} |\mathcal{H}|_{\infty,S}(\nu') d\nu' \right) |r^{(3-\frac{2}{p})} \mathcal{L}_O^{N-1} \psi|_{p,S}(\lambda, \nu_*) \quad (14.70)$$

$$+ C \left(\int_{\nu}^{\nu_*} d\nu' e^{c \left(\int_{\nu}^{\nu_*} |\mathcal{H}|_{\infty,S}(\nu') d\nu' - \int_{\nu'}^{\nu_*} |\mathcal{H}|_{\infty,S}(\nu'') d\nu'' \right)} (2|\hat{\chi}|_{\infty,S} + |\text{tr}\chi|_{\infty,S}) |r^{3-\frac{2}{p}} (\mathcal{L}_O^{N-1} \beta)|_{p,S} \right. \\ \left. + \int_{\nu}^{\nu_*} d\nu' e^{c \left(\int_{\nu}^{\nu_*} |\mathcal{H}|_{\infty,S}(\nu') d\nu' - \int_{\nu'}^{\nu_*} |\mathcal{H}|_{\infty,S}(\nu'') d\nu'' \right)} \left[|r^{3-\frac{2}{p}} \{(good)_{1+4}\}|_{p,S} \right] (\lambda, \nu') \right).$$

We start proving that the recursive assumption is satisfied considering only the $\{(good)_1\}$ term. Let us consider, therefore, the following simplified inequality,

$$|r^{(3-\frac{2}{p})} \mathcal{L}_O^{N-1} \psi|_{p,S}(\lambda, \nu) \leq \left(\exp c \int_{\nu}^{\nu_*} |\mathcal{H}|_{\infty,S}(\nu') d\nu' \right) |r^{(3-\frac{2}{p})} \mathcal{L}_O^{N-1} \psi|_{p,S}(\lambda, \nu_*) \\ + C \left(\int_{\nu}^{\nu_*} d\nu' e^{c \left(\int_{\nu}^{\nu_*} |\mathcal{H}|_{\infty,S}(\nu') d\nu' - \int_{\nu'}^{\nu_*} |\mathcal{H}|_{\infty,S}(\nu'') d\nu'' \right)} |r^{(3-\frac{2}{p})} \{(good)_1\}|_{p,S} \right) \quad (14.71) \\ \leq e^{\Gamma_*(\nu)} |r^{(3-\frac{2}{p})} \mathcal{L}_O^{N-1} \psi|_{p,S}(\lambda, \nu_*) + C \left(\int_{\nu}^{\nu_*} d\nu' e^{(\Gamma_*(\nu) - \Gamma_*(\nu'))} |r^{3-\frac{2}{p}} \{(good)_1\}|_{p,S} \right)$$

where, with $\tilde{C} > 0$,

$$\Gamma_*(\nu) \equiv c \int_{\nu}^{\nu_*} |\mathcal{H}|_{\infty,S}(\nu') d\nu' = \tilde{C} \frac{\nu_* - \nu}{\nu_* \nu}. \quad (14.72)$$

Let us estimate now $|r^{3-\frac{2}{p}} \{(good)_1\}|_{p,S}$, subsequently we will use this estimate as paradigmatic of all the estimates involving a product of two or more derived connection coefficients, with order of derivatives less than N in order to apply the inductive estimates,

$$|r^{3-\frac{2}{p}} \{(good)_1\}|_{p,S} \leq 2 \sum_{k=1}^{N-1} \binom{N-1}{k} |r^{3-\frac{2}{p}} (\mathcal{L}_O^{N-1-k} (\nabla \hat{\chi}) (\mathcal{L}_O^k \hat{\chi}))| \quad (14.73) \\ - \frac{3}{2} \sum_{q=1}^{N-1} \binom{N-1}{q} |r^{3-\frac{2}{p}} (\hat{\mathcal{L}}_O^q \Omega^2 U) (\hat{\mathcal{L}}_O^{N-1-q} \psi)| + |r^{3-\frac{2}{p}} \sum_{k=0}^{N-1} \binom{N-1}{k} (\mathcal{L}_O^{N-1-k} |\hat{\chi}|^2) (\mathcal{L}_O^k \eta)| \\ + \sum_{q=1}^{N-1} \binom{N-1}{q} |r^{3-\frac{2}{p}} (\mathcal{L}_O^q (\Omega \hat{\chi})) (\mathcal{L}_O^{N-1-q} \psi)| + \sum_{q=0}^{N-2} \binom{N-1}{q} |r^{3-\frac{2}{p}} (\mathcal{L}_O^{N-2-q} E) (\mathcal{L}_O^q \psi)|$$

$$- \sum_{q=0}^{N-1} \binom{N-1}{q} |r^{3-\frac{2}{p}}(\mathcal{L}_O^q(\Omega U)(\mathcal{L}_O^{N-1-q}\hat{\chi}\eta))| + \sum_{q=1}^{N-1} \binom{N-1}{q} |r^{3-\frac{2}{p}}(\mathcal{L}_O^{N-2-q}\beta)(\mathcal{L}_O^q \text{tr}\chi)|$$

To apply the inductive assumptions to this expression we need a recursive assumption also for Ω . We assume, see the estimate 11.2,¹¹³

$$|\mathcal{L}_O^J \Omega|_{p,S} \leq C_4 e^{((J-1)-2)\delta} e^{(J-1)\Gamma} \frac{(J-1)!}{(J-1)^\alpha} \frac{1}{\rho_{0,1}^{(J-1)}}. \quad (14.74)$$

Let us start looking at the first sum of 14.73, if $N-1$ is even,

$$\begin{aligned} & \sum_{k=1}^{N-1} \binom{N-1}{k} |r^{3-\frac{2}{p}}(\mathcal{L}_O^{N-1-k})(\nabla \hat{\chi})(\mathcal{L}_O^k \hat{\chi})| \leq \\ & \frac{1}{r^2} \left[\sum_{k=1}^{\frac{N-1}{2}} \binom{N}{k} |r^3 \mathcal{L}_O^k \nabla \hat{\chi}|_\infty |r^{2-\frac{2}{p}} \mathcal{L}_O^{N-1-k} \hat{\chi}|_{p,S} + \sum_{k>\frac{N-1}{2}} \binom{N}{k} |r^{3-\frac{2}{p}} \mathcal{L}_O^k \nabla \hat{\chi}|_{p,S} |r^2 \mathcal{L}_O^{N-1-k} \hat{\chi}|_\infty \right] \end{aligned} \quad (14.75)$$

In order to estimate inductively the right hand sides of these equations we recall the inductive assumptions on the right hand side terms

$$\begin{aligned} |r^3 \mathcal{L}_O^k \nabla \hat{\chi}|_{\infty,S} & \leq C e^{k\delta} e^{k\Gamma(\lambda)} \frac{(k+2)!}{(k+2)^\alpha} \frac{1}{\rho_{0,1}^{k+2}} \\ |r^{2-\frac{2}{p}} \mathcal{L}_O^{N-1-k} \hat{\chi}|_{p,S} & \leq C e^{((N-3-k)\delta} e^{(N-3-k)\Gamma(\lambda)} \frac{(N-1-k)!}{(N-1-k)^\alpha} \frac{1}{\rho_{0,1}^{(N-1-k)}} \\ |r^{3-\frac{2}{p}} \mathcal{L}_O^k \nabla \hat{\chi}|_{p,S} & \leq C e^{(k-1)\delta} e^{(k-1)\Gamma(\lambda)} \frac{(k+1)!}{(k+1)^\alpha} \frac{1}{\rho_{0,1}^{k+1}} \\ |r^2 \mathcal{L}_O^{N-1-k} \hat{\chi}|_{\infty,S} & \leq C e^{(N-1-k)\delta} e^{(N-1-k)\Gamma(\lambda)} \frac{(N+1-k)!}{(N+1-k)^\alpha} \frac{1}{\rho_{0,1}^{N+1-k}}. \end{aligned} \quad (14.76)$$

Therefore,

$$\begin{aligned} & \frac{1}{r^2} \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \binom{N}{k} |r^3 \mathcal{L}_O^k \nabla \hat{\chi}|_\infty |r^{2-\frac{2}{p}} \mathcal{L}_O^{N-1-k} \hat{\chi}|_{p,S} \\ & \leq \frac{C^2}{r^2} \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \binom{N}{k} e^{k\delta} e^{k\Gamma(\lambda)} \frac{(k+2)!}{(k+2)^\alpha} \frac{1}{\rho_{0,1}^{k+2}} e^{(N-1-k)\delta} e^{(N-1-k)\Gamma(\lambda)} \frac{(N+1-k)!}{(N+1-k)^\alpha} \frac{1}{\rho_{0,1}^{N+1-k}} \\ & \leq \left(C \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \right) \left[\sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} C N^\alpha \frac{(k+2)!}{k!} \frac{e^{-\delta} e^{-\Gamma(\lambda)}}{(k+2)^\alpha (N+1-k)^\alpha} \frac{1}{\rho_{0,1}^2} \right] \end{aligned}$$

¹¹³For $J > 0$ the decay of $\mathcal{L}_O^J \Omega$ is better.

$$\begin{aligned}
&\leq \frac{1}{r^2} \left(C \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \right) \left[\frac{N^\alpha}{\left(\frac{N}{2}\right)^\alpha} \sum_{k=1}^{\left[\frac{N}{2}\right]} C \frac{(k+2)!}{k!} \frac{e^{-\delta} e^{-\Gamma(\lambda)}}{(k+2)^\alpha} \frac{1}{\rho_{0,1}^2} \right] \\
&\leq \frac{1}{r^2} \left(C \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \right) \left[\frac{C 2^\alpha e^{-\delta} e^{-\Gamma(\lambda)}}{\rho_{0,1}^2} \sum_{k=1}^{\left[\frac{N}{2}\right]} \frac{1}{(k+2)^{\alpha-2}} \right] \\
&\leq \frac{1}{r^2} \left(C C e^{-\delta} \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \right) \leq \frac{1}{r^2} \left(C e^{-\delta} \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \right) \quad (14.77)
\end{aligned}$$

with

$$\left[\frac{C 2^\alpha e^{-\delta} e^{-\Gamma(\lambda)}}{\rho_{0,1}^2} \sum_{k=1}^{\left[\frac{N}{2}\right]} \frac{1}{(k+2)^{\alpha-2}} \right] \leq C e^{-\delta}$$

and $\alpha > 3$. Clearly the term

$$\sum_{k > \frac{N-1}{2}}^{N-1} \binom{N}{k} |r^{3-\frac{2}{p}} \mathcal{L}_O^k \nabla \hat{\chi}|_{p,S} |r^2 \mathcal{L}_O^{N-1-k} \hat{\chi}|_{\infty,S}$$

Can be estimated in the same way. Therefore, considering only this contribution from $\{(good)_1\}$, it follows, recalling the last slice estimates of the connection coefficients, see subsection 9.6,

$$\begin{aligned}
&|r^{3-\frac{2}{p}} \mathcal{L}_O^{N-1} \psi|_{p,S}(\lambda, \nu) \leq e^{\Gamma_*(\nu)} |r^{(3-\frac{2}{p})} \mathcal{L}_O^{N-1} \psi|_{p,S}(\lambda, \nu_*) \\
&\quad + C \left(\int_{\nu}^{\nu_*} d\nu' e^{(\Gamma_*(\nu) - \Gamma_*(\nu'))} |r^{3-\frac{2}{p}} \{(good)_1\}|_{p,S} \right) \\
&\leq C_0^{(0)} e^{\Gamma_*(\nu)} \left(e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} \right) \quad (14.78)
\end{aligned}$$

$$\begin{aligned}
&+ \left(C \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \right) \left(C e^{-\delta} \int_{\nu}^{\nu_*} d\nu' \frac{e^{(\Gamma_*(\nu) - \Gamma_*(\nu'))}}{r^2} \right) \\
&\leq C_0^{(0)} e^{\Gamma_*(\nu)} \left(e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} \right) \quad (14.79)
\end{aligned}$$

$$\begin{aligned}
&+ \left(C \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \right) \left(C e^{-\delta} \int_{\nu}^{\nu_*} \frac{d\nu'}{\nu'^2} e^{(\Gamma_*(\nu) - \Gamma_*(\nu'))} \right) \\
&\leq \left(C \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \right) \left(\frac{C_0^{(0)}}{C} e^{(\Gamma_*(\nu))} \right. \\
&\quad \left. + C e^{-\delta} \int_{\nu}^{\nu_*} \frac{d\nu'}{\nu'^2} e^{(\Gamma_*(\nu) - \Gamma_*(\nu'))} \right)
\end{aligned}$$

$$\leq \left(C \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \right), \quad (14.80)$$

choosing suitable $C > C_0$ and δ sufficiently large.

The other terms in $\{good\}_1$, and $\{good\}_4$ can be estimated in the same way, for the sake of completeness we only consider another term, namely

$$\sum_{k=1}^{N-2} \binom{N-1}{k} |r^{3-\frac{2}{p}}(\mathcal{L}_O^{k-1}E) \cdot (\mathcal{L}_O^{N-1-k}\psi)|_{p,S} \quad (14.81)$$

recalling 14.31 we write, a bit symbolically,

$$\mathcal{L}_O^{k-1}E \text{ " = " } \mathcal{L}_O^{k-1}\hat{\chi}\underline{\eta} + \mathcal{L}_O^{k-1}\text{tr}\chi\underline{\eta} + \mathcal{L}_O^{k-1}\hat{\chi}\underline{\eta} + \mathcal{L}_O^{k-1}\text{tr}\chi\underline{\eta} + \nabla^{k-1}\Psi + \mathcal{L}_O^{k-1}\mathcal{H} \quad (14.82)$$

where with Ψ we denote a generic null Riemann component. We consider the contribution of the first term, the ones associated to the other term will be of the same kind, but easier and the one associated to the last term will be examined later on. We have

$$\begin{aligned} & \sum_{k=1}^{N-2} \binom{N-1}{k} |r^{3-\frac{2}{p}}(\mathcal{L}_O^{k-1}\hat{\chi}\underline{\eta}) \cdot (\mathcal{L}_O^{N-1-k}\psi)|_{p,S} \\ &= \frac{1}{r^4} \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \binom{N-1}{k} |r^2(\mathcal{L}_O^{k-1}\hat{\chi}\underline{\eta})|_\infty |r^{3-\frac{2}{p}}(\mathcal{L}_O^{N-1-k}\psi)|_{p,S} \\ &+ \frac{1}{r^4} \sum_{k=\lfloor \frac{N}{2} \rfloor+1}^{N-2} \binom{N-1}{k} |r^2(\mathcal{L}_O^{N-1-k}\psi)|_\infty |r^{3-\frac{2}{p}}(\mathcal{L}_O^{k-1}\hat{\chi}\underline{\eta})|_{p,S} \\ &\leq \frac{1}{r^4} \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \sum_{q=0}^{k-1} \binom{N-1}{k} \binom{k-1}{q} |r^3\mathcal{L}_O^q\hat{\chi}|_\infty |r^2\mathcal{L}_O^{k-1-q}\underline{\eta}|_\infty |r^{3-\frac{2}{p}}\mathcal{L}_O^{N-1-k}\psi|_{p,S} \\ &+ \frac{1}{r^4} \sum_{k=\lfloor \frac{N}{2} \rfloor+1}^{N-2} \sum_{q=0}^{k-1} \binom{N-1}{k} \binom{k-1}{q} |r^2(\mathcal{L}_O^{N-1-k}\psi)|_\infty |r^2\nabla^q\hat{\chi}|_\infty |r^{3-\frac{2}{p}}\mathcal{L}_O^{k-1-q}\underline{\eta}|_{p,S} \end{aligned} \quad (14.83)$$

Let us estimate the first term of the last sum in 14.83,

$$\frac{1}{r^4} \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \sum_{q=0}^{k-1} \binom{N-1}{k} \binom{k-1}{q} |r^2\mathcal{L}_O^q\hat{\chi}|_\infty |r^2\mathcal{L}_O^{k-1-q}\underline{\eta}|_\infty |r^{(3-\frac{2}{p})}\mathcal{L}_O^{N-1-k}\psi|_{p,S} \quad (14.84)$$

Using the inductive assumptions:

$$\begin{aligned}
&\leq \frac{C^3 N!}{r^4} \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \frac{(k-1)!}{(N-k-1)!(k+1)!} \sum_{q=0}^{k-1} \frac{1}{(k-1-q)!q!} \left[\frac{(q+1)!}{(q+1)^\alpha} \frac{e^{(q-1)\delta} e^{(q-1)\Gamma(\lambda)}}{\rho_{0,1}^{q+1}} \right] \\
&\cdot \left[\frac{(k-q)!}{(k-q)^\alpha} \frac{e^{(k-q-2)\delta} e^{(k-q-2)\Gamma(\lambda)}}{\rho_{0,1}^{(k-q)}} \right] \frac{(N-k)!}{(N-k)^\alpha} \frac{e^{(N-k-2)\delta} e^{(N-k-2)\Gamma(\lambda)}}{\rho_{0,1}^{N-k}} \\
&\leq \frac{C^3}{r^4} \frac{N!}{N^\alpha} \frac{e^{(N-5)\delta} e^{(N-5)\Gamma(\lambda)}}{\rho_{0,1}^N} \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \frac{(N-k)}{Nk(k+1)} \sum_{q=0}^{k-1} \left[\frac{1}{\rho_{0,1}} \frac{(q+1)}{(q+1)^\alpha} \right] \\
&\cdot \left[\frac{(k-q)}{(k-q)^\alpha} \right] \frac{N^\alpha}{\lfloor \frac{N}{2} \rfloor^\alpha} \\
&\leq \frac{1}{r^4} \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)}}{\rho_{0,1}^N} \right) e^{-3\delta} e^{-3\Gamma(\lambda)} 2^\alpha \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{k(k+1)} \sum_{q=0}^{k-1} \left[\frac{1}{\rho_{0,1}} \frac{(q+1)}{(q+1)^\alpha} \right] \\
&\cdot \left[\frac{(k-q)}{(k-q)^\alpha} \right] \\
&\leq \frac{C^3}{r^4} \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)}}{\rho_{0,1}^N} \right) \left(e^{-3\delta} e^{-3\Gamma(\lambda)} 2^\alpha C \right) \tag{14.85}
\end{aligned}$$

where

$$\sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{k(k+1)} \sum_{q=0}^{k-1} \left[\frac{1}{\rho_{0,1}} \frac{(q+1)}{(q+1)^\alpha} \right] \cdot \left[\frac{(k-q)}{(k-q)^\alpha} \right] \leq C \tag{14.86}$$

Choosing δ sufficiently large we obtain

$$\sum_{k=1}^{N-2} \binom{N-1}{k} \left| r^{3-\frac{2}{p}} (\mathcal{L}_O^{k-1} E) \cdot (\mathcal{L}_O^{N-1-k} \psi) \right|_{p,S} \leq \frac{1}{r^4} \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)}}{\rho_{0,1}^N} \right) \tag{14.87}$$

From the estimate 14.87 it follows that this contribution to the integral of 14.81 is similar to the previous terms.

At last let us control the following term of the integral in the inequality 14.70,

$$\int_{\nu}^{\nu_*} d\nu' e^{(\Gamma_*(\nu) - \Gamma_*(\nu'))} (|2\hat{\chi}|_{\infty,S} + |\text{tr}\chi|_{\infty,S}) |r^{(3-\frac{2}{p})} \mathcal{L}_O^{N-1} \beta|_{p,S} . \tag{14.88}$$

To estimate it we need to control $|r^{3-\frac{2}{p}} \mathcal{L}_O^{N-1} \beta|_{p,S}$, this is the content of Theorem 4.1. From it the following estimate holds for the \mathcal{L}_O^{N-1} angular derivatives of

the Riemann tensor null component β :

$$|r^{\frac{7}{2}+(N-1)-\frac{2}{p}} \mathcal{L}_O^{N-1} \beta|_{p,S} \leq C^{(1)} e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} \quad (14.89)$$

and the integral 14.136 satisfies:

$$\begin{aligned} & \int_\nu^{\nu_*} d\nu' e^{(\Gamma_*(\nu)-\Gamma_*(\nu'))} (|2\hat{\chi}|_{\infty,S} + |\text{tr}\chi|_{\infty,S}) |r^{(3-\frac{2}{p})} \mathcal{L}_O^{N-1} \beta|_{p,S} \\ & \leq \epsilon \left(\int_\nu^{\nu_*} \frac{d\nu'}{\nu' r^{\frac{1}{2}}} e^{(\Gamma_*(\nu)-\Gamma_*(\nu'))} |r^{(\frac{7}{2}-\frac{2}{p})} \mathcal{L}_O^{N-1} \beta|_{p,S}(\nu') \right. \\ & \quad \left. + \int_\nu^{\nu_*} \frac{d\nu'}{\nu' r^{\frac{3}{2}}} e^{(\Gamma_*(\nu)-\Gamma_*(\nu'))} |r^{\frac{7}{2}-\frac{2}{p}} \mathcal{L}_O^{N-1} \beta|_{p,S}(\nu') \right) \\ & \leq c C^{(1)} e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} \left(\int_\nu^{\nu_*} \frac{d\nu'}{\nu' r^{\frac{1}{2}}} e^{(\Gamma_*(\nu)-\Gamma_*(\nu'))} \right. \\ & \quad \left. + \int_\nu^{\nu_*} \frac{d\nu'}{r^{\frac{1}{2}} \nu'^2} e^{(\Gamma_*(\nu)-\Gamma_*(\nu'))} \right) \\ & \leq C_0 e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N}, \end{aligned}$$

Remark:

Observe that these estimates are consistent due to the fact that the null Riemann components can be bounded by a constant $C^{(1)}$ smaller than the constants bounding the connection coefficients.

Looking at the remaining contributions to the integral in the right hand side of 14.70 it is easy to realise that they all have the same structure and can be estimated in the same way. Therefore, collecting all these estimates together it follows that Theorem 9.1 is proved choosing δ sufficiently large.

14.4 Proof of Lemma 9.4

Applying \mathcal{L}_O^{N-1} to the transport equation 9.38,¹¹⁴ and using the result of Lemma 9.1, we obtain an equation with the same structure as the transport equation 9.2 for $\mathcal{L}_O^{N-1} \tilde{\psi}$,

$$\frac{\mathcal{D}(\mathcal{L}_O^{N-1} \tilde{\mu})}{\partial \nu} + \Omega \text{tr}\chi \cdot (\mathcal{L}_O^{N-1} \tilde{\mu}) + \mathcal{L}_O^{N-1} \left[(\Omega \tilde{F} - \overline{\Omega \tilde{F}}) + (\Omega \tilde{H} - \overline{\Omega \tilde{H}}) \right] = \{(\widetilde{good})_1\} \quad (14.90)$$

With, see equation 14.48

¹¹⁴We apply ∇^{N-1} instead of ∇^N as $\tilde{\mu}$ depends on $\text{d}\text{iv}\eta$ and, therefore, the control of $\mathcal{L}_O^{N-1} \tilde{\mu}$ allows to control $\mathcal{L}_O^N \eta$.

$$\{(\widetilde{good})_1\} = [\mathcal{D}_\nu, \mathcal{L}_O^{N-1}] \tilde{\mu} = \sum_{q=0}^{N-2} \binom{N-1}{q} (\mathcal{L}_O^{N-2-q} E) (\mathcal{L}_O^q \tilde{\mu}) \quad (14.91)$$

Rewriting $\mathcal{L}_O^{N-1}[(\Omega \tilde{F} - \overline{\Omega \tilde{F}}) + (\Omega \tilde{H} - \overline{\Omega \tilde{H}})]$, extracting the higher derivative terms,

$$\mathcal{L}_O^{N-1}[(\Omega \tilde{F} - \overline{\Omega \tilde{F}}) + (\Omega \tilde{H} - \overline{\Omega \tilde{H}})] = \Omega \hat{\chi} \cdot \mathcal{L}_O^{N-1}(\nabla \hat{\otimes} \eta) - 2\Omega \eta \cdot \mathcal{L}_O^{N-1} \beta + \Omega \text{tr} \chi \mathcal{L}_O^{N-1} \rho + \{(\widetilde{good})_2\}, \quad (14.92)$$

we obtain the transport equation 9.40.

14.5 Proof of Lemma 9.5

We start from the following Hodge system

$$\begin{aligned} d\text{iv} \eta &= -\tilde{\mu} + \frac{1}{2}(\chi \underline{\chi} - \overline{\chi \underline{\chi}}) - (\rho - \bar{\rho}) \equiv F^{(0)} \\ \text{curl} \eta &= \sigma - \frac{1}{2} \hat{\chi} \wedge \hat{\chi} \equiv G^{(0)}. \end{aligned}$$

Following the steps of lemma 9.3 we can write,

$$\begin{aligned} d\text{iv} \mathcal{L}_O^{N-1} \eta &= [d\text{iv}, \mathcal{L}_O^{N-1}] \eta + \mathcal{L}_O^{N-1}(-\tilde{\mu} + \frac{1}{2}(\chi \underline{\chi} - \overline{\chi \underline{\chi}}) - (\rho - \bar{\rho})) \\ &= -\mathcal{L}_O^{N-1} \tilde{\mu} - \mathcal{L}_O^{N-1} \rho + \{(\widetilde{good})_{3one}\} \end{aligned} \quad (14.93)$$

With

$$\begin{aligned} \{(\widetilde{good})_{3one}\} &= -\mathcal{L}_O^{N-1} \frac{1}{2}(\chi \underline{\chi} - \overline{\chi \underline{\chi}}) + \mathcal{L}_O^{N-2} \rho + [d\text{iv}, \mathcal{L}_O^{N-1}] \eta \\ &= -\mathcal{L}_O^{N-1}(\frac{1}{2}(\chi \underline{\chi} - \overline{\chi \underline{\chi}}) + \mathcal{L}_O^{N-2} \bar{\rho} + \sum_{q=0}^{N-2} (\binom{N-1}{q} (\mathcal{L}_O^{N-2-q} \overline{E}) (\mathcal{L}_O^q \eta) \\ &\quad + \binom{N-1}{q} (\mathcal{L}_O^{N-2-q} \overline{\mathcal{H}}) (\mathcal{L}_O^q \nabla \eta)) \end{aligned}$$

In the same way we can write

$$\begin{aligned} \text{curl} \mathcal{L}_O^{N-1} \eta &= [\text{curl}, \mathcal{L}_O^{N-1}] \eta + \mathcal{L}_O^{N-1}(\sigma - \frac{1}{2} \hat{\chi} \wedge \hat{\chi}) \\ &= -\mathcal{L}_O^{N-1} \sigma + \{(\widetilde{good})_{3bis}\} \end{aligned} \quad (14.94)$$

With

$$\{(\widetilde{good})_{3bis}\} = \mathcal{L}_O^{N-1}(\sigma - \frac{1}{2}\hat{\underline{\chi}} \wedge \hat{\chi}) + [\text{curl}, \mathcal{L}_O^{N-1}]\eta \quad (14.95)$$

Now by proposition 4.1.3 of [Kl-Ni2] we have that, defining

$$\{(\widetilde{good})_3\} = \{(\widetilde{good})_{3one}\} + \{(\widetilde{good})_{3bis}\} \quad (14.96)$$

$$|\nabla \mathcal{L}_O^{N-1}\eta|_{p,S} \leq c \left(|\mathcal{L}_O^{N-1}\tilde{\mu}|_{p,S} + |\mathcal{L}_O^{N-1}\rho|_{p,S} + |\mathcal{L}_O^{N-1}\sigma|_{p,S} + |\{(\widetilde{good})_3\}|_{p,S} \right) \quad (14.97)$$

With c a suitable constant. Consequently, following the same strategy of lemma 9.3 the same inequality holds for $|\mathcal{L}_O^{N-1}\nabla\eta|_{p,S}$ and consequently for $|\mathcal{L}_O^{N-1}\nabla\hat{\otimes}\eta|_{p,S}$, hence we obtain

$$|\mathcal{L}_O\tilde{\mu}|_{p,S} \leq c \left(|\mathcal{L}_O^{N-1}\tilde{\mu}|_{p,S} + |\mathcal{L}_O^{N-1}\rho|_{p,S} + |\mathcal{L}_O^{N-1}\sigma|_{p,S} + |\{(\widetilde{good})_3\}|_{p,S} \right) \quad (14.98)$$

Substituting this last inequality in equation 14.92

$$\begin{aligned} \left| \frac{\partial(r^{2-\frac{2}{p}}\mathcal{L}_O^{N-1}\tilde{\mu})}{\partial\nu} \right|_{p,S} &\leq C \left(|\{(\widetilde{good})_1\}|_{p,S} + |\{(\widetilde{good})_2\}|_{p,S} \right. \\ &\quad \left. + |\Omega\hat{\chi}\{(\widetilde{good})_3\}|_{p,S} + |\Omega\eta\mathcal{L}_O^{N-1}\beta|_{p,S} + |\mathcal{L}_O^{N-1}\rho|_{p,S} \right) \end{aligned} \quad (14.99)$$

Now performing the same steps of theorem 9.1 we obtain inequality 9.5 and proof of lemma 9.5 is achieved.

14.6 Proof of Lemma 9.6

We prove first the following

Lemma 14.1 *The structure equations, see (I), imply that \tilde{V} satisfies the following transport equation,*

$$\begin{aligned} \mathbf{D}_4\tilde{V} + \frac{\text{tr}\chi}{2}\tilde{V} + \hat{\chi} \cdot \tilde{V} + 4\omega\tilde{V} &= \Omega(\nabla\rho) - \Omega(\nabla\log\Omega)\rho + \nabla[\Omega(\underline{\eta} \cdot \eta - 2\zeta^2 - 2\zeta \cdot \nabla\log\Omega)] \\ &\quad + [4\Omega(\eta + \underline{\eta})\omega\underline{\omega}] . \end{aligned} \quad (14.100)$$

Proof of Lemma 14.1: We start computing $\mathbf{D}_4 \tilde{V}$:

$$\begin{aligned}
\mathbf{D}_4 \tilde{V} &= \mathbf{D}_4 \nabla (\Omega \mathbf{D}_3 \log \Omega) = \mathbf{D}_4 [(\nabla \Omega) \mathbf{D}_3 \log \Omega + \Omega (\nabla \mathbf{D}_3 \log \Omega)] \\
&= (\mathbf{D}_4 \nabla \Omega) (\mathbf{D}_3 \log \Omega) + (\nabla \Omega) (\mathbf{D}_4 \mathbf{D}_3 \log \Omega) + (\mathbf{D}_4 \Omega) (\nabla \mathbf{D}_3 \log \Omega) + \Omega \mathbf{D}_4 (\nabla \mathbf{D}_3 \log \Omega) \\
&= (\mathbf{D}_4 \nabla \Omega) (\mathbf{D}_3 \log \Omega) + (\nabla \Omega) (\mathbf{D}_4 \mathbf{D}_3 \log \Omega) + (\mathbf{D}_4 \Omega) (\nabla \mathbf{D}_3 \log \Omega) + \Omega \nabla \mathbf{D}_4 \mathbf{D}_3 \log \Omega \\
&\quad + \Omega [\mathbf{D}_4, \nabla] \mathbf{D}_3 \log \Omega \tag{14.101} \\
&= (\mathbf{D}_4 \nabla \Omega) (\mathbf{D}_3 \log \Omega) + (\nabla \Omega) (\mathbf{D}_4 \mathbf{D}_3 \log \Omega) + (\mathbf{D}_4 \Omega) (\nabla \mathbf{D}_3 \log \Omega) + \Omega \nabla \mathbf{D}_4 \mathbf{D}_3 \log \Omega \\
&\quad + \Omega (\nabla \log \Omega) \mathbf{D}_4 \mathbf{D}_3 \log \Omega - \Omega \chi \cdot \nabla \mathbf{D}_3 \log \Omega \\
&= (\mathbf{D}_4 \nabla \Omega) (\mathbf{D}_3 \log \Omega) + 2(\nabla \Omega) (\mathbf{D}_4 \mathbf{D}_3 \log \Omega) + (\mathbf{D}_4 \Omega) (\nabla \mathbf{D}_3 \log \Omega) + \Omega \nabla \mathbf{D}_4 \mathbf{D}_3 \log \Omega \\
&\quad - \Omega \chi \cdot \nabla \mathbf{D}_3 \log \Omega \\
&= (\mathbf{D}_4 \nabla \Omega) (\mathbf{D}_3 \log \Omega) + 2(\nabla \Omega) (\mathbf{D}_4 \mathbf{D}_3 \log \Omega) + (\mathbf{D}_4 \Omega) (\nabla \mathbf{D}_3 \log \Omega) + \Omega \nabla \mathbf{D}_4 \mathbf{D}_3 \log \Omega \\
&\quad - \chi \cdot \nabla (\Omega \mathbf{D}_3 \log \Omega) + \chi \cdot (\nabla \Omega) \mathbf{D}_3 \log \Omega \\
&= (\mathbf{D}_4 \nabla \Omega) (\mathbf{D}_3 \log \Omega) + 2(\nabla \Omega) (\mathbf{D}_4 \mathbf{D}_3 \log \Omega) + (\mathbf{D}_4 \Omega) (\nabla \mathbf{D}_3 \log \Omega) + \Omega \nabla \mathbf{D}_4 \mathbf{D}_3 \log \Omega \\
&\quad - \chi \cdot \tilde{V} + \Omega \chi \cdot (\nabla \log \Omega) \mathbf{D}_3 \log \Omega .
\end{aligned}$$

Summarizing

$$\begin{aligned}
\mathbf{D}_4 \tilde{V} &= (\mathbf{D}_4 \nabla \Omega) (\mathbf{D}_3 \log \Omega) + 2(\nabla \Omega) (\mathbf{D}_4 \mathbf{D}_3 \log \Omega) + (\mathbf{D}_4 \Omega) (\nabla \mathbf{D}_3 \log \Omega) + \Omega \nabla \mathbf{D}_4 \mathbf{D}_3 \log \Omega \\
&\quad - \chi \cdot \tilde{V} + \Omega \chi \cdot (\nabla \log \Omega) \mathbf{D}_3 \log \Omega \\
&= (\nabla \mathbf{D}_4 \Omega + [\mathbf{D}_4, \nabla] \Omega) (\mathbf{D}_3 \log \Omega) + 2(\nabla \Omega) (\mathbf{D}_4 \mathbf{D}_3 \log \Omega) + (\mathbf{D}_4 \Omega) (\nabla \mathbf{D}_3 \log \Omega) \\
&\quad + \Omega \nabla (\mathbf{D}_4 \mathbf{D}_3 \log \Omega) - \chi \cdot \left(\tilde{V} - \Omega (\nabla \log \Omega) \mathbf{D}_3 \log \Omega \right) \\
&= -\chi \cdot \tilde{V} + \Omega \nabla (\mathbf{D}_4 \mathbf{D}_3 \log \Omega) + (\nabla \mathbf{D}_4 \Omega) (\mathbf{D}_3 \log \Omega) + \Omega (\mathbf{D}_4 \log \Omega) (\nabla \mathbf{D}_3 \log \Omega) + 2(\nabla \Omega) (\mathbf{D}_4 \mathbf{D}_3 \log \Omega) \\
&\quad + \left[((\nabla \log \Omega) \mathbf{D}_4 \Omega - \chi \cdot \nabla \Omega) (\mathbf{D}_3 \log \Omega) + \Omega \chi \cdot (\nabla \log \Omega) (\mathbf{D}_3 \log \Omega) \right] \\
&= -\chi \cdot \tilde{V} + \Omega \nabla (\mathbf{D}_4 \mathbf{D}_3 \log \Omega) + (\mathbf{D}_3 \log \Omega) (\nabla \Omega \mathbf{D}_4 \log \Omega) + (\mathbf{D}_4 \log \Omega) (\nabla \Omega \mathbf{D}_3 \log \Omega) \\
&\quad + 2\Omega (\nabla \log \Omega) (\mathbf{D}_4 \mathbf{D}_3 \log \Omega) + \left[-\Omega (\mathbf{D}_4 \log \Omega) (\nabla \log \Omega) (\mathbf{D}_3 \log \Omega) + (\Omega (\nabla \log \Omega) (\mathbf{D}_4 \log \Omega) \right. \\
&\quad \left. - \Omega \chi \cdot (\nabla \log \Omega) (\mathbf{D}_3 \log \Omega) + \Omega \chi \cdot (\nabla \log \Omega) (\mathbf{D}_3 \log \Omega) \right] \tag{14.102}
\end{aligned}$$

We reexpress now the term $\Omega \nabla (\mathbf{D}_4 \mathbf{D}_3 \log \Omega)$ in the following way: recall that these equations hold

$$\begin{aligned}
\mathbf{D}_3 \mathbf{D}_4 \log \Omega + \mathbf{D}_4 \mathbf{D}_3 \log \Omega &= -2(\mathbf{D}_3 \log \Omega) (\mathbf{D}_4 \log \Omega) + 2(\underline{\eta} \cdot \eta - 2\zeta^2 - \rho) \\
\mathbf{D}_4 \mathbf{D}_3 \log \Omega - \mathbf{D}_3 \mathbf{D}_4 \log \Omega &= -4\zeta \cdot \nabla \log \Omega \tag{14.103}
\end{aligned}$$

which imply

$$\mathbf{D}_4 \mathbf{D}_3 \log \Omega = -(\mathbf{D}_3 \log \Omega) (\mathbf{D}_4 \log \Omega) + (\underline{\eta} \cdot \eta - 2\zeta^2 - 2\zeta \cdot \nabla \log \Omega) - \rho . \tag{14.104}$$

Therefore

$$\Omega \nabla (\mathbf{D}_4 \mathbf{D}_3 \log \Omega) = -\Omega \nabla (\mathbf{D}_3 \log \Omega \mathbf{D}_4 \log \Omega) + \Omega \nabla (\underline{\eta} \cdot \eta - 2\zeta^2 - 2\zeta \cdot \nabla \log \Omega) - \Omega \nabla \rho$$

$$\begin{aligned}
&= -\Omega(\nabla \mathbf{D}_3 \log \Omega)(\mathbf{D}_4 \log \Omega) - (\mathbf{D}_3 \log \Omega)\Omega(\nabla \mathbf{D}_4 \log \Omega) - \Omega \nabla \rho + \Omega \nabla(\underline{\eta} \cdot \eta - 2\zeta^2 - 2\zeta \cdot \nabla \log \Omega) \\
&= -(\nabla \Omega \mathbf{D}_3 \log \Omega)(\mathbf{D}_4 \log \Omega) - (\mathbf{D}_3 \log \Omega)(\nabla \Omega \mathbf{D}_4 \log \Omega) + 2\Omega(\nabla \log \Omega)(\mathbf{D}_3 \log \Omega)(\mathbf{D}_4 \log \Omega) \\
&\quad - \Omega \nabla \rho + \Omega \nabla(\underline{\eta} \cdot \eta - 2\zeta^2 - 2\zeta \cdot \nabla \log \Omega) \\
&= (\mathbf{D}_4 \log \Omega)\tilde{V} - (\mathbf{D}_3 \log \Omega)(\nabla \Omega \mathbf{D}_4 \log \Omega) - \Omega \nabla \rho + \Omega \nabla(\underline{\eta} \cdot \eta - 2\zeta^2 - 2\zeta \cdot \nabla \log \Omega) \\
&\quad + 2\Omega(\nabla \log \Omega)(\mathbf{D}_3 \log \Omega)(\mathbf{D}_4 \log \Omega) .
\end{aligned} \tag{14.105}$$

Substituting this expression in 14.102 we obtain

$$\begin{aligned}
\mathbf{D}_4 \tilde{V} &= -\chi \cdot \tilde{V} + 2(\mathbf{D}_4 \log \Omega)\tilde{V} - \Omega \nabla \rho + \Omega \nabla(\underline{\eta} \cdot \eta - 2\zeta^2 - 2\zeta \cdot \nabla \log \Omega) + 2\Omega(\nabla \log \Omega)(\mathbf{D}_4 \mathbf{D}_3 \log \Omega) \\
&\quad + \left[2\Omega(\nabla \log \Omega)(\mathbf{D}_3 \log \Omega)(\mathbf{D}_4 \log \Omega) - \Omega(\mathbf{D}_4 \log \Omega)(\nabla \log \Omega)(\mathbf{D}_3 \log \Omega) \right. \\
&\quad \left. + (\Omega(\nabla \log \Omega)(\mathbf{D}_4 \log \Omega) - \Omega \chi \cdot (\nabla \log \Omega))(\mathbf{D}_3 \log \Omega) + \Omega \chi \cdot (\nabla \log \Omega)(\mathbf{D}_3 \log \Omega) \right] \\
&= -\chi \cdot \tilde{V} + 2(\mathbf{D}_4 \log \Omega)\tilde{V} - \nabla(\Omega \rho) + \nabla[\Omega(\underline{\eta} \cdot \eta - 2\zeta^2 - 2\zeta \cdot \nabla \log \Omega)] \\
&\quad + \left[2\Omega(\nabla \log \Omega)(\mathbf{D}_3 \log \Omega)(\mathbf{D}_4 \log \Omega) \right] ,
\end{aligned} \tag{14.106}$$

which, recalling the definition of ω and $\underline{\omega}$ becomes

$$\mathbf{D}_4 \tilde{V} = -\chi \cdot \tilde{V} - 4\omega \tilde{V} - \nabla(\Omega \rho) + \nabla[\Omega(\underline{\eta} \cdot \eta - 2\zeta^2 - 2\zeta \cdot \nabla \log \Omega)] + 4\Omega(\eta + \underline{\eta})\omega \underline{\omega} \tag{14.107}$$

which we rewrite as

$$\begin{aligned}
\mathbf{D}_4 \tilde{V} + \frac{\text{tr} \chi}{2} \tilde{V} + \hat{\chi} \cdot \tilde{V} + 4\omega \tilde{V} &= \\
&= -\Omega(\nabla \rho) - \Omega(\nabla \log \Omega)\rho + \nabla[\Omega(\underline{\eta} \cdot \eta - 2\zeta^2 - 2\zeta \cdot \nabla \log \Omega)] + [4\Omega(\eta + \underline{\eta})\omega \underline{\omega}] .
\end{aligned} \tag{14.108}$$

This equation has the positive characteristic that it does not depend on \underline{V} (exactly as the equation for $\tilde{\mu}$ does not depend on for $\underline{\mu}$). Still, due to the presence of $\Omega \nabla \rho$, there is a loss of derivative which we have to cure. To face this problem we look at the transport equation of $\underline{\omega} = \text{div}(\tilde{V} - \Omega \underline{\beta})$.

We start looking at the transport equation of $\mathbf{D}_4 \nabla \tilde{V}$,

$$\begin{aligned}
\mathbf{D}_4 \nabla_a \tilde{V}_b &= \nabla_a \mathbf{D}_4 \tilde{V}_b + [\mathbf{D}_4, \nabla_a] \tilde{V}_b \\
&= \nabla_a \mathbf{D}_4 \tilde{V}_b + \left[-\chi_{ac} \nabla_c \tilde{V}_b - \underline{\eta}_b \chi_{ac} \tilde{V}_c + \chi_{ab}(\underline{\eta} \cdot \tilde{V}) + (\nabla_a \log \Omega) \mathbf{D}_4 \tilde{V}_b + ([D_\tau, D_\rho] \tilde{V}_\sigma) e_4^\tau e_a^\rho e_b^\sigma \right] \\
&= -\nabla_a(\chi_{bc} \cdot \tilde{V}_c) - 4\nabla_a(\omega \tilde{V}_b) + \nabla_a[-\Omega(\nabla_b \rho) - \Omega(\nabla_b \log \Omega)\rho + \nabla_b[\Omega(\underline{\eta} \cdot \eta - 2\zeta^2 - 2\zeta \cdot \nabla \log \Omega)]] \\
&\quad + \nabla_a[4\Omega(\eta + \underline{\eta})\omega \underline{\omega}] + \left[-\chi_{ac} \nabla_c \tilde{V}_b - \underline{\eta}_b \chi_{ac} \tilde{V}_c + \chi_{ab}(\underline{\eta} \cdot \tilde{V}) + (\nabla_a \log \Omega) \mathbf{D}_4 \tilde{V}_b + ([D_\tau, D_\rho] \tilde{V}_\sigma) e_4^\tau e_a^\rho e_b^\sigma \right] \\
&= -\text{tr} \chi \nabla_a \tilde{V}_b - (\hat{\chi}_{bc} \nabla_a \tilde{V}_c + \hat{\chi}_{ac} \nabla_b \tilde{V}_c) - 4\omega \nabla_a \tilde{V}_b - \Omega \nabla_a \nabla_b \rho + \Omega(\nabla_a \log \Omega) \nabla_b \rho - \Omega(\nabla_b \log \Omega) \nabla_a \rho \\
&\quad + (\nabla_a \log \Omega) \mathbf{D}_4 \tilde{V}_b + \left[-(\nabla_a \chi_{bc}) \cdot \tilde{V}_c - 4(\nabla_a \omega) \tilde{V}_b - \Omega(\nabla_a \nabla_b \log \Omega)\rho + \nabla_a \nabla_b [\Omega(\underline{\eta} \cdot \eta - 2\zeta^2 - 2\zeta \cdot \nabla \log \Omega)] \right] \\
&\quad + \Omega(\nabla_a \log \Omega)(\nabla_b \log \Omega)\rho + \nabla_a[4\Omega(\eta + \underline{\eta})\omega \underline{\omega}] + \left[-\underline{\eta}_b \chi_{ac} \tilde{V}_c + \chi_{ab}(\underline{\eta} \cdot \tilde{V}) + ([D_\tau, D_\rho] \tilde{V}_\sigma) e_4^\tau e_a^\rho e_b^\sigma \right]
\end{aligned}$$

$$\begin{aligned}
&= -\text{tr}\chi \nabla_a \tilde{V}_b - (\hat{\chi}_{bc} \nabla_a \tilde{V}_c + \hat{\chi}_{ac} \nabla_c \tilde{V}_b) - 4\omega \nabla_a \tilde{V}_b - \Omega \nabla_a \nabla_b \rho + 2\Omega (\nabla_a \log \Omega) \nabla_b \rho - \Omega (\nabla_b \log \Omega) \nabla_a \rho \\
&\quad + (\nabla_a \log \Omega) \left[-\chi_{bc} \tilde{V}_c - 4\omega \tilde{V}_b - \Omega (\nabla_b \log \Omega) \rho + \nabla_b [\Omega (\underline{\eta} \cdot \eta - 2\zeta^2 - 2\zeta \cdot \nabla \log \Omega)] + 4\Omega (\eta + \underline{\eta})_b \omega \underline{\omega} \right] \\
&\quad + \left[-(\nabla_a \chi_{bc}) \cdot \tilde{V}_c - 4(\nabla_a \omega) \tilde{V}_b - \Omega (\nabla_a \nabla_b \log \Omega) \rho + \nabla_a \nabla_b [\Omega (\underline{\eta} \cdot \eta - 2\zeta^2 - 2\zeta \cdot \nabla \log \Omega)] \right] \\
&\quad + \Omega (\nabla_a \log \Omega) (\nabla_b \log \Omega) \rho + \nabla_a [4\Omega (\eta + \underline{\eta}) \omega \underline{\omega}] + \left[-\underline{\eta}_b \chi_{ac} \tilde{V}_c + \chi_{ab} (\underline{\eta} \cdot \tilde{V}) + \left([D_\tau, D_\rho] \tilde{V}_\sigma \right) e_4^\tau e_a^\rho e_b^\sigma \right]
\end{aligned}$$

which we rewrite ordering the r.h.s terms following the number of derivatives (of the metric components)

$$\begin{aligned}
&\mathbf{D}_4 \nabla_a \tilde{V}_b + \text{tr}\chi \nabla_a \tilde{V}_b + (\hat{\chi}_{bc} \nabla_a \tilde{V}_c + \hat{\chi}_{ac} \nabla_c \tilde{V}_b) + 4\omega \nabla_a \tilde{V}_b = -\Omega \nabla_a \nabla_b \rho \\
&\quad + 2\Omega (\nabla_a \log \Omega) \nabla_b \rho - \Omega (\nabla_b \log \Omega) \nabla_a \rho + \mathcal{Q}_{ab}
\end{aligned} \tag{14.109}$$

where \mathcal{Q}_{ab} collects all the terms which depend at most on two derivatives or on three derivatives of terms whose estimates have been already obtained as, for instance, the second derivatives of ζ ,

$$\begin{aligned}
\mathcal{Q}_{ab} &= (\nabla_a \log \Omega) \left[-\chi_{bc} \tilde{V}_c - 4\omega \tilde{V}_b - \Omega (\nabla_b \log \Omega) \rho + \nabla_b [\Omega (\underline{\eta} \cdot \eta - 2\zeta^2 - 2\zeta \cdot \nabla \log \Omega)] + 4\Omega (\eta + \underline{\eta})_b \omega \underline{\omega} \right] \\
&\quad + \left[-(\nabla_a \chi_{bc}) \cdot \tilde{V}_c - 4(\nabla_a \omega) \tilde{V}_b - \Omega (\nabla_a \nabla_b \log \Omega) \rho + \nabla_a \nabla_b [\Omega (\underline{\eta} \cdot \eta - 2\zeta^2 - 2\zeta \cdot \nabla \log \Omega)] \right] \\
&\quad + \Omega (\nabla_a \log \Omega) (\nabla_b \log \Omega) \rho + \nabla_a [4\Omega (\eta + \underline{\eta}) \omega \underline{\omega}] + \left[-\underline{\eta}_b \chi_{ac} \tilde{V}_c + \chi_{ab} (\underline{\eta} \cdot \tilde{V}) + \left([D_\tau, D_\rho] \tilde{V}_\sigma \right) e_4^\tau e_a^\rho e_b^\sigma \right]
\end{aligned}$$

Taking the trace we obtain

$$\mathbf{D}_4 \text{div} \tilde{V} + \text{tr}\chi \text{div} \tilde{V} + 2\hat{\chi} \cdot \nabla \tilde{V} + 4\omega \text{div} \tilde{V} = -\Omega \Delta \rho + \Omega (\nabla \log \Omega) \cdot \nabla \rho + \text{tr} \mathcal{Q} \tag{14.110}$$

As expected the term $-\Omega \Delta \rho$ implies a loss of derivatives. To cure it we look at the transport equation for $\text{div} \underline{\beta}$. From eq. (4.4.5) of [Kl-Ni2] we have,

$$\mathbf{D}_4 (\Omega \nabla_a \underline{\beta}_b) + \text{tr}\chi \Omega \nabla_a \underline{\beta}_b = -\Omega \nabla_a \nabla_b \rho + \Omega \nabla_a {}^* \nabla_b \sigma - \frac{1}{2} \text{tr}\chi \Omega \nabla_a \underline{\beta}_b - \hat{\chi}_{ac} \Omega \nabla_c \beta_b + \mathcal{H} , \tag{14.111}$$

$$\mathbf{D}_4 (\Omega \text{div} \underline{\beta}) + \text{tr}\chi (\Omega \text{div} \underline{\beta}) = -\Omega \Delta \rho - \frac{1}{2} \text{tr}\chi \Omega \text{div} \underline{\beta} - \hat{\chi} \cdot \Omega \nabla \underline{\beta} + \text{tr} \mathcal{H} .$$

Moreover

$$\mathbf{D}_4 \text{div} (\Omega \underline{\beta}) = \mathbf{D}_4 (\Omega \text{div} \underline{\beta}) + \mathbf{D}_4 (\nabla \Omega \cdot \underline{\beta}) \tag{14.112}$$

and

$$\begin{aligned}
\mathbf{D}_4 \nabla \Omega &= [\mathbf{D}_4, \nabla] \Omega + \nabla \mathbf{D}_4 \Omega = [\mathbf{D}_4, \nabla] \Omega + \nabla \Omega \mathbf{D}_4 \log \Omega \\
&= [\mathbf{D}_4, \nabla] \Omega - 2\nabla \Omega \omega = [\mathbf{D}_4, \nabla] \Omega - 2\Omega (\nabla \log \Omega) \omega - 2\Omega \nabla \omega \\
&= [\mathbf{D}_4, \nabla] \Omega - \Omega (\eta + \underline{\eta}) \omega - 2\Omega \nabla \omega \\
&= (\zeta + \underline{\eta}) \mathbf{D}_4 \Omega - \chi \cdot \nabla \Omega - \Omega (\eta + \underline{\eta}) \omega - 2\Omega \nabla \omega \\
&= -2\Omega (\zeta + \underline{\eta}) \omega - \Omega \chi \cdot (\zeta + \underline{\eta}) - \Omega (\eta + \underline{\eta}) \omega - 2\Omega \nabla \omega \\
&= -4\Omega (\nabla \log \Omega) \omega - \Omega \chi \cdot (\nabla \log \Omega) - 2\Omega \nabla \omega
\end{aligned} \tag{14.113}$$

therefore

$$(\mathbf{D}_4 \nabla \Omega) \cdot \underline{\beta} = - \left[(4\Omega\omega(\nabla \log \Omega) + \Omega(\nabla \log \Omega) \cdot \chi) + 2\Omega \nabla \omega \right] \cdot \underline{\beta} \quad (14.114)$$

moreover

$$\nabla \Omega \cdot \mathbf{D}_4 \underline{\beta} + \nabla \Omega \text{tr} \chi \underline{\beta} = \nabla \Omega \cdot [-\nabla \rho + [\cdot \cdot \cdot]] \quad (14.115)$$

it follows

$$\begin{aligned} \mathbf{D}_4 \text{div}(\Omega \underline{\beta}) + \text{tr} \chi \text{div}(\Omega \underline{\beta}) &= \mathbf{D}_4(\Omega \text{div} \underline{\beta}) + (\mathbf{D}_4 \nabla \Omega) \cdot \underline{\beta} + \nabla \Omega \cdot \mathbf{D}_4 \underline{\beta} + \text{tr} \chi(\Omega \text{div} \underline{\beta}) + (\nabla \Omega) \text{tr} \chi \underline{\beta} \\ &= (\mathbf{D}_4(\Omega \text{div} \underline{\beta}) + \text{tr} \chi(\Omega \text{div} \underline{\beta})) + (\nabla \Omega) \cdot (\mathbf{D}_4 \underline{\beta} + \text{tr} \chi \underline{\beta}) + (\mathbf{D}_4 \nabla \Omega) \cdot \underline{\beta} \\ &= \left[-\Omega \underline{\Delta} \rho - \frac{1}{2} \text{tr} \chi \Omega \text{div} \underline{\beta} - \hat{\chi} \cdot \Omega \nabla \beta + \text{tr} \mathcal{H} \right] + (\nabla \Omega) [-\nabla \rho + [\cdot \cdot \cdot]] - \left[(4\Omega\omega(\nabla \log \Omega) + \Omega(\nabla \log \Omega) \cdot \chi) + 2\Omega \nabla \omega \right] \cdot \underline{\beta} \\ &= \left[-\Omega \underline{\Delta} \rho - \frac{1}{2} \text{tr} \chi \Omega \text{div} \underline{\beta} - \hat{\chi} \cdot \Omega \nabla \beta + \text{tr} \mathcal{H} \right] + (\nabla \Omega) [-\nabla \rho + (2\omega \underline{\beta} + 2\hat{\chi} \cdot \beta + {}^* \nabla \sigma - 3(\underline{\eta} \rho - {}^* \underline{\eta} \sigma))] \\ &\quad - \left[(4\Omega\omega(\nabla \log \Omega) + \Omega(\nabla \log \Omega) \cdot \chi) + 2\Omega \nabla \omega \right] \cdot \underline{\beta} \end{aligned}$$

which implies

$$\mathbf{D}_4[\text{div}(\tilde{V} - \Omega \underline{\beta})] + \text{tr} \chi[\text{div}(\tilde{V} - \Omega \underline{\beta})] \quad (14.116)$$

$$\begin{aligned} &= [-2\hat{\chi} \cdot \nabla \tilde{V} - 4\omega \text{div} \tilde{V} + \Omega(\nabla \log \Omega) \cdot \nabla \rho + \text{tr} \mathcal{Q}] - \left[-\frac{1}{2} \text{tr} \chi \Omega \text{div} \underline{\beta} - \hat{\chi} \cdot \Omega \nabla \beta + \text{tr} \mathcal{H} \right] \\ &\quad + (\nabla \Omega) [-\nabla \rho + (2\omega \underline{\beta} + 2\hat{\chi} \cdot \beta + {}^* \nabla \sigma - 3(\underline{\eta} \rho - {}^* \underline{\eta} \sigma))] + \left[(4\Omega\omega(\nabla \log \Omega) + \Omega(\nabla \log \Omega) \cdot \chi) + 2\Omega \nabla \omega \right] \cdot \underline{\beta} \\ &= -4\omega[\text{div}(\tilde{V} - \Omega \underline{\beta})] - 2\hat{\chi} \cdot \nabla \tilde{V} \quad (14.117) \end{aligned}$$

$$\begin{aligned} &\quad + \left[-4\omega \text{div}(\Omega \underline{\beta}) + \Omega(\nabla \log \Omega) \cdot \nabla \rho + \frac{1}{2} \text{tr} \chi \Omega \text{div} \underline{\beta} + \hat{\chi} \cdot \Omega \nabla \beta + (\nabla \Omega)(-\nabla \rho + {}^* \nabla \sigma) \right] \\ &\quad + [\text{tr} \mathcal{Q} + \text{tr} \mathcal{H} + (\nabla \Omega)(2\omega \underline{\beta} + 2\hat{\chi} \cdot \beta - 3(\underline{\eta} \rho - {}^* \underline{\eta})) + (4\Omega\omega(\nabla \log \Omega) + \Omega(\nabla \log \Omega) \cdot \chi) \cdot \underline{\beta} + 2\Omega \nabla \omega \cdot \underline{\beta}] \end{aligned}$$

which we rewrite in a more compact way

$$\begin{aligned} \mathbf{D}_4 \text{div} \hat{V} + \text{tr} \chi \text{div} \hat{V} &= -4\omega \text{div} \hat{V} - 2\hat{\chi} \cdot \nabla \hat{V} \\ &\quad + \left[-2\hat{\chi} \cdot \nabla \Omega \underline{\beta} - 4\omega \text{div}(\Omega \underline{\beta}) + \Omega(\nabla \log \Omega) \cdot \nabla \rho + \frac{1}{2} \text{tr} \chi \Omega \text{div} \underline{\beta} + \hat{\chi} \cdot \Omega \nabla \beta + (\nabla \Omega)(-\nabla \rho + {}^* \nabla \sigma) \right] \\ &\quad + \Omega^{-1} \{ \mathcal{F} \} \quad (14.118) \end{aligned}$$

where $\Omega^{-1} \{ \mathcal{F} \}$ collect all lower order terms which can be estimated using the inductive assumptions, more precisely

$$\Omega^{-1} \{ \mathcal{F} \} = \text{tr}(\mathcal{Q} + \mathcal{H}) + [(\nabla \Omega)(2\omega \underline{\beta} + 2\hat{\chi} \cdot \beta - 3(\underline{\eta} \rho - {}^* \underline{\eta})) + (4\Omega\omega(\nabla \log \Omega) + \Omega(\nabla \log \Omega) \cdot \chi) \cdot \underline{\beta} + 2\Omega \nabla \omega \cdot \underline{\beta}] \quad (14.119)$$

Therefore

$$\begin{aligned} \frac{d\tilde{\omega}}{d\nu} + \Omega \text{tr} \chi \tilde{\omega} + 4\Omega \omega \tilde{\omega} = & -\Omega \left[2\hat{\chi} \cdot \nabla \hat{V} - 2\hat{\chi} \cdot \nabla \Omega \underline{\beta} - 4\omega \text{d}\text{iv}(\Omega \underline{\beta}) + \Omega(\nabla \log \Omega) \cdot \nabla \rho \right. \\ & \left. + \frac{1}{2} \text{tr} \chi \Omega \text{d}\text{iv} \underline{\beta} + \hat{\chi} \cdot \Omega \nabla \beta + (\nabla \Omega)(-\nabla \rho + \star \nabla \sigma) \right] + \{\mathcal{F}\}. \end{aligned} \quad (14.120)$$

To prove the transport equation 9.55 for $\mathcal{L}_O^{N-2} \tilde{\omega}$ we proceed exactly in the same way as we did for $\mathcal{L}_O^{N-1} \psi$. Starting from

$$\mathcal{L}_O^{N-2} \frac{\mathcal{D} \tilde{\omega}}{\partial \nu} = \frac{\mathcal{D}}{\partial \nu} (\mathcal{L}_O^{N-2} \tilde{\omega}) + [\mathcal{L}_O^{N-2}, \frac{\mathcal{D}}{\partial \nu}] \tilde{\omega} \quad (14.121)$$

and using 9.21 it follows

$$\frac{\mathcal{D}}{\partial \nu} (\mathcal{L}_O^{N-2} \tilde{\omega}) = \mathcal{L}_O^{N-2} \frac{\mathcal{D} \tilde{\omega}}{\partial \nu} - \sum_{q=0}^{N-2} \binom{N-2}{q} (\mathcal{L}_O^{N-3-q} E)(\mathcal{L}_O^q \tilde{\omega}) \quad (14.122)$$

From equation 14.120 it follows:

$$\begin{aligned} \mathcal{L}_O^{N-2} \frac{\mathcal{D} \tilde{\omega}}{\partial \nu} = & -\mathcal{L}_O^{N-2} ((\Omega \text{tr} \chi + 4\Omega \omega) \tilde{\omega}) - 2\mathcal{L}_O^{N-2} \Omega \hat{\chi} \cdot \nabla \hat{V} \\ & + \mathcal{L}_O^{N-2} \left\{ \Omega \left[-2\hat{\chi} \cdot \nabla \Omega \underline{\beta} - 4\omega \text{d}\text{iv}(\Omega \underline{\beta}) + \Omega(\nabla \log \Omega) \cdot \nabla \rho + \frac{1}{2} \text{tr} \chi \Omega \text{d}\text{iv} \underline{\beta} + \hat{\chi} \cdot \Omega \nabla \beta + (\nabla \Omega)(-\nabla \rho + \star \nabla \sigma) \right] \right\} \\ & + \mathcal{L}_O^{N-2} \{\mathcal{F}\} \\ = & -(\Omega \text{tr} \chi + 4\Omega) \mathcal{L}_O^{N-2} \tilde{\omega} - 2\Omega \hat{\chi} \cdot \nabla \hat{V} + \Omega \left[-2\Omega \hat{\chi} \cdot \mathcal{L}_O^{N-2} \nabla \underline{\beta} - 4\Omega \omega \mathcal{L}_O^{N-2} \text{d}\text{iv} \underline{\beta} + \Omega(\nabla \log \Omega) \cdot \mathcal{L}_O^{N-2} \nabla \rho \right. \\ & \left. + \frac{1}{2} \text{tr} \chi \Omega \mathcal{L}_O^{N-2} \text{d}\text{iv} \underline{\beta} + \hat{\chi} \cdot \Omega \mathcal{L}_O^{N-2} \nabla \beta + (\nabla \Omega)(-\mathcal{L}_O^{N-2} \nabla \rho + \mathcal{L}_O^{N-2} \star \nabla \sigma) \right] + \{\mathcal{G}\} + \mathcal{L}_O^{N-2} \{\mathcal{F}\}, \end{aligned} \quad (14.123)$$

where

$$\begin{aligned} \{\mathcal{G}\} = & \mathcal{L}_O^{N-2} \left\{ \Omega \left[-2\hat{\chi} \cdot \nabla \Omega \underline{\beta} - 4\omega \text{d}\text{iv}(\Omega \underline{\beta}) + \Omega(\nabla \log \Omega) \cdot \nabla \rho + \frac{1}{2} \text{tr} \chi \Omega \text{d}\text{iv} \underline{\beta} + \hat{\chi} \cdot \Omega \nabla \beta + (\nabla \Omega)(-\nabla \rho + \star \nabla \sigma) \right] \right\} \\ & - \Omega \left[-2\Omega \hat{\chi} \cdot \mathcal{L}_O^{N-2} \nabla \underline{\beta} - 4\Omega \omega \mathcal{L}_O^{N-2} \text{d}\text{iv} \underline{\beta} + \Omega(\nabla \log \Omega) \cdot \mathcal{L}_O^{N-2} \nabla \rho \right. \\ & \left. + \frac{1}{2} \text{tr} \chi \Omega \mathcal{L}_O^{N-2} \text{d}\text{iv} \underline{\beta} + \hat{\chi} \cdot \Omega \mathcal{L}_O^{N-2} \nabla \beta + (\nabla \Omega)(-\mathcal{L}_O^{N-2} \nabla \rho + \mathcal{L}_O^{N-2} \star \nabla \sigma) \right] \\ & - \left[2\mathcal{L}_O^{N-2} \Omega \hat{\chi} \cdot \nabla \hat{V} - 2\Omega \hat{\chi} \cdot \mathcal{L}_O^{N-1} \hat{V} - (\Omega \text{tr} \chi + 4\Omega) \mathcal{L}_O^{N-2} \tilde{\omega} - 2\Omega \hat{\chi} \cdot \mathcal{L}_O^{N-2} \nabla \hat{V} \right]. \end{aligned} \quad (14.124)$$

It is clear that the terms $\{\mathcal{G}\}$ and $\{\tilde{\mathcal{F}}\} \equiv \mathcal{L}_O^{N-2} \{\mathcal{F}\}$ do not contain the highest derivatives and can be estimated inductively in the standard way, the higher order derivatives terms in 14.123 are the more important ones. Collecting all the terms we have for $\mathcal{L}_O^{N-2} \tilde{\omega}$ the following transport equation,

$$\frac{\mathcal{D}}{\partial \nu} (\mathcal{L}_O^{N-2} \tilde{\omega}) + (\Omega \text{tr} \chi + 4\Omega \omega) (\mathcal{L}_O^{N-2} \tilde{\omega}) + (\Omega \chi) \cdot (\mathcal{L}_O^{N-2} \tilde{\omega})$$

$$\begin{aligned}
&= -2\Omega\hat{\chi}\mathcal{L}_O^{N-2}\nabla\hat{V} + \Omega[-2\Omega\hat{\chi}\cdot\mathcal{L}_O^{N-1}\underline{\beta} - 4\Omega\omega\mathcal{L}_O^{N-2}\mathfrak{d}\nabla\underline{\beta} + \Omega(\nabla\log\Omega)\cdot\mathcal{L}_O^{N-1}\rho \\
&\quad + \frac{1}{2}\text{tr}\chi\Omega\mathcal{L}_O^{N-2}\mathfrak{d}\nabla\underline{\beta} + \hat{\chi}\cdot\Omega\mathcal{L}_O^{N-1}\beta + (\nabla\Omega)(-\mathcal{L}_O^{N-1}\rho + \mathcal{L}_O^{N-2}\star\nabla\sigma)] \\
&\quad + \{\mathcal{L}\} + \{\mathcal{G}\} + \{\tilde{\mathcal{F}}\}, \tag{14.125}
\end{aligned}$$

where

$$\{\tilde{\mathcal{F}}\} = \mathcal{L}_O^{N-2}\{\mathcal{F}\} \tag{14.126}$$

$$\{\mathcal{L}\} = -\sum_{q=0}^{N-2} \binom{N-2}{q} (\mathcal{L}_O^{N-3-q}E)(\mathcal{L}_O^q\underline{\omega}) \tag{14.127}$$

14.7 Proof of Lemma 9.7

As done previously for lemma 9.5, we start from the following Hodge system

$$\begin{aligned}
\mathfrak{d}\nabla\hat{V} &= \underline{\omega} \equiv F^{(0)} \\
\text{curl}\hat{V} &= \Omega(\nabla\log\Omega\wedge\underline{\beta}) + \Omega\text{curl}\underline{\beta} \equiv G^{(0)} \tag{14.128}
\end{aligned}$$

and we proceed exactly as in the proof of Lemma 9.5 with the substitution of η with $\hat{V} = 2\nabla\Omega\underline{\omega} - \Omega\underline{\beta}$. As \hat{V} is at the level of a Riemann component (a derivative of a connection coefficient) we have to control its \mathcal{L}_O^{N-1} angular derivative. We repeat the first steps of the previous lemma with the quoted substitutions and obtain.

$$|\nabla\mathcal{L}_O^{N-2}\hat{V}|_{p,S} \leq |\widetilde{\{\text{good}_3\}}|_{p,S} + |\tilde{\omega}|_{p,S} \tag{14.129}$$

with

$$|\widetilde{\{\text{good}_3\}}|_{p,S} = |[\mathfrak{d}\nabla, \mathcal{L}_O^{N-2}]\hat{V}|_{p,S} + |[\text{curl}, \mathcal{L}_O^{N-2}]\hat{V}|_{p,S} + \mathcal{L}_O^{N-1}(\Omega(\nabla\log\Omega\wedge\underline{\beta}) + \Omega\text{curl}\underline{\beta}) \tag{14.130}$$

Are terms which can be easily estimated by the inductive assumptions.
Hence

$$|\mathcal{L}_O^{N-2}\nabla\hat{V}|_{p,S} \leq |\widetilde{\{\text{good}_3\}}|_{p,S} + |\tilde{\omega}|_{p,S} \tag{14.131}$$

14.8 The estimates for the underlined connection coefficients

The estimates for the underlined connection coefficients can be substantially obtained in the same way as done for the non underlined ones, the only difference

occurs for the integral estimates of the terms classified as $\{(\underline{good})_I\}$, let us calculate in details the corresponding estimates in this case, defining

$$\tilde{\Gamma}(\lambda) \equiv c \int_{\lambda_0}^{\lambda} |\mathcal{H}|_{\infty, S}(\lambda') d\lambda' = \tilde{C} \frac{\lambda - \lambda_0}{\lambda \lambda'}. \quad (14.132)$$

$$\begin{aligned} |r^{3-\frac{2}{p}} \mathcal{L}_O^{N-1} \underline{\Psi}|_{p, S}(\lambda, \nu) &\leq e^{\tilde{\Gamma}(\lambda)} |r^{(3-\frac{2}{p})} \mathcal{L}_O^{N-1} \underline{\Psi}|_{p, S}(\lambda_0, \nu) \\ &\quad + C \left(\int_{\lambda_0}^{\lambda} d\lambda' e^{(\tilde{\Gamma}(\lambda_0) - \tilde{\Gamma}(\lambda'))} |r^{3-\frac{2}{p}} \{(\underline{good})_I\}|_{p, S} \right) \\ &\leq C_0^{(0)} e^{\tilde{\Gamma}(\lambda)} \left(e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} \right) \end{aligned} \quad (14.133)$$

$$\begin{aligned} &+ \left(C \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \right) \left(C e^{-\delta} \int_{\lambda_0}^{\lambda} d\lambda' \frac{e^{(-(N-2)\Gamma(\lambda) + \tilde{\Gamma}(\lambda) - \tilde{\Gamma}(\lambda'))}}{r^2} \right) \\ &\leq C_0^{(0)} e^{\tilde{\Gamma}(\lambda)} \left(e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} \right) \end{aligned} \quad (14.134)$$

$$\begin{aligned} &+ \left(C \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \right) \left(C e^{-\delta} \int_{\lambda_0}^{\lambda} d\lambda' \frac{e^{(-(N-2)\Gamma(\lambda') + \tilde{\Gamma}(\lambda) - \tilde{\Gamma}(\lambda'))}}{r^2} \right) \\ &\leq \left(C \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \right) \left(\frac{C_0^{(0)}}{C} e^{\tilde{\Gamma}(\lambda)} \right. \\ &\quad \left. + C' e^{-\delta} \int_{\lambda_0}^{\lambda} \frac{d\lambda'}{\lambda'^2} e^{-(N-2)C \frac{\lambda' - \lambda_0}{\lambda' \lambda_0}} \right) \\ &\leq \left(C'' \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \right) \end{aligned} \quad (14.135)$$

Choosing suitably $C_0^{(0)} < C$ and δ and were the last integral has been estimated as in equation 14.80. At las let us estimate the corresponding of integral 14.92

$$\int_{\lambda_0}^{\lambda} d\lambda' e^{(\tilde{\Gamma}(\lambda) - \tilde{\Gamma}(\lambda'))} (|2\hat{\chi}|_{\infty, S} + |\text{tr}\chi|_{\infty, S}) |r^{(3-\frac{2}{p})} \mathcal{L}_O^{N-1} \underline{\beta}|_{p, S}. \quad (14.136)$$

In order to estimate this term we can perform the same steps of the previous term, with the difference that the decay of the $\text{tr}\chi$ is worst. Actually we obtain

$$\begin{aligned} &\int_{\lambda_0}^{\lambda} d\lambda' e^{(\tilde{\Gamma}(\lambda) - \tilde{\Gamma}(\lambda'))} (|2\hat{\chi}|_{\infty, S} + |\text{tr}\chi|_{\infty, S}) |r^{(3-\frac{2}{p})} \mathcal{L}_O^{N-1} \underline{\beta}|_{p, S}. \quad (14.137) \\ &\leq + C' e^{-\delta} \lambda^{\frac{1}{2}} \int_{\lambda_0}^{\lambda} \frac{d\lambda'}{\lambda'^2} e^{-(N-2)C \frac{\lambda' - \lambda_0}{\lambda' \lambda_0}} \leq \left(C'' \frac{N!}{N^\alpha} \frac{1}{\rho_{0,1}^N} e^{(N-2)\delta} e^{(N-2)\Gamma(\lambda)} \right) \end{aligned} \quad (14.138)$$

14.9 Proof of corollary 9.5

Once we have the estimate for $\mathcal{L}_O^{N-1}\psi = \mathcal{L}_O^{N-1}\nabla U$, we have to estimate $\mathcal{L}_O^N U$. To do this we have to exploit a modified version of proposition 3.3.3 and 3.3.4 of [Kl-Ni2], let us recall it.

Proposition 14.1 *Let U_α be 1-form, then it does exists a constant C such that*

$$|\nabla U|^2 \leq C \sum_i |\mathcal{L}_{(i)} O U|^2 \quad (14.139)$$

Let $U_{\alpha\beta}$ be a symmetric two tensor, then it does exists a constant C such that

$$|\nabla U|^2 \leq C \sum_i |\mathcal{L}_{(i)} O U|^2 \quad (14.140)$$

We add to this proposition the

Proposition 14.2 *Let U_α be 1-form, then it does exists a constant C such that*

$$|\nabla U|^4 \leq C \sum_i |\mathcal{L}_{(i)} O U|^4 \quad (14.141)$$

Let $U_{\alpha\beta}$ be a symmetric two tensor, then it does exists a constant C such that

$$|\nabla U|^4 \leq C \sum_i |\mathcal{L}_{(i)} O U|^4 \quad (14.142)$$

The proof is a straightforward modification of the propositions 3.3.3 and 3.3.4 of [Ch-Kl].

Then, exploiting the previous propositions, we obtain

$$|\mathcal{L}_O^{N-1}\psi| = |\mathcal{L}_O^{N-1}\nabla U|_{p,S} = |\nabla \mathcal{L}_O U| + |[\mathcal{L}_O^{N-1}, \nabla]U|_{p,S} \leq C|\mathcal{L}_O^N U|_{p,S} + \widetilde{good_6} \quad (14.143)$$

Hence we have the right estimates for $|\mathcal{L}_O^N U|_{p,S}$ with $p = 2, 4$. The other values of p are obtained by interpolation. Clearly the same estimate can be done for $\mathcal{L}_O^{N-1}\nabla \chi$.

14.10 Proof of corollary 9.9

In the following we prove corollary 9.9, to do it we proceed inductively in the following way, we observe that

$$\begin{aligned}\mathcal{L}_O^J S_{AC} &= \mathcal{L}_O^J \mathbf{g}(e_A, \nabla e_C) = \sum_{k=0}^J \binom{J}{k} \mathbf{g}(\mathcal{L}_O^k e_A, \mathcal{L}_O^{J-k} \nabla e_C) = \sum_{k=0}^J \binom{J}{k} \gamma_{ac} \mathcal{L}_O^k e_A^a \mathcal{L}_O^{J-k} \nabla e_C^c \\ &= \sum_{k=0}^J \binom{J}{k} \sum_E \theta_a^E \gamma_{ac} \mathcal{L}_O^k e_A^a \theta_c^E \mathcal{L}_O^{J-k} \nabla e_C^c = \sum_E \sum_{k=0}^J \binom{J}{k} \mathbf{g}(e_E \mathcal{L}_O^k e_A) \mathbf{g}(e_E \mathcal{L}_O^{J-k} \nabla e_C) .\end{aligned}$$

Therefore

$$\mathbf{g}(e_A, \mathcal{L}_O^J \nabla e_C) = \mathcal{L}_O^J S - \sum_E \sum_{k=1}^J \binom{J}{k} \mathbf{g}(e_E \mathcal{L}_O^k e_A) \mathbf{g}(e_E \mathcal{L}_O^{J-k} \nabla e_C) \quad (14.144)$$

and

$$\begin{aligned}& |r^{(J+1-\frac{2}{p})} \mathbf{g}(e_A, \mathcal{L}_O^J \nabla e_C)|_{p,S} \\ & \leq |r^{(J+1-\frac{2}{p})} \mathcal{L}_O^J S_{AC}|_{p,S} + \sum_E \sum_{k=1}^J \binom{J}{k} |r^{(J+1-\frac{2}{p})} \mathbf{g}(e_E, \mathcal{L}_O^k e_A) \mathbf{g}(e_E, \mathcal{L}_O^{J-k} \nabla e_C)|_{p,S} \\ & \leq |r^{(J+1-\frac{2}{p})} \mathcal{L}_O^J S_{AC}|_{p,S} + \sum_E \sum_{k=1}^{\lfloor \frac{J}{2} \rfloor} \binom{J}{k} |r^k \mathbf{g}(e_E, \mathcal{L}_O^k e_A)|_{\infty,S} |r^{(J+1-k-\frac{2}{p})} \mathbf{g}(e_E, \mathcal{L}_O^{J-k} \nabla e_C)|_{p,S} \\ & \quad + \sum_E \sum_{k=\lfloor \frac{J}{2} \rfloor+1}^J \binom{J}{k} |r^{(k-\frac{2}{p})} \mathbf{g}(e_E, \mathcal{L}_O^k e_A)|_{p,S} |r^{(J+1-k)} \mathbf{g}(e_E, \mathcal{L}_O^{J-k} \nabla e_C)|_{\infty,S} .\end{aligned} \quad (14.145)$$

We make the inductive assumptions for $|r^{(k-\frac{2}{p})} \mathbf{g}(e_E, \mathcal{L}_O^k e_A)|_{p,S}$, with $k \leq J$

$$|r^{(k-\frac{2}{p})} \mathbf{g}(e_E, \mathcal{L}_O^k e_A)|_{p,S} \leq c \left(\frac{(k-1)!}{(k-1)^\alpha} \frac{e^{(k-3)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^{k-1}} \right) \quad (14.146)$$

therefore

$$\begin{aligned}& |r^{(J+1-\frac{2}{p})} \mathbf{g}(e_A, \mathcal{L}_O^J \nabla e_C)|_{p,S} \leq c \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right) \\ & + \sum_E \sum_{k=1}^{\lfloor \frac{J}{2} \rfloor} \binom{J}{k} c^2 \left(\frac{k!}{k^\alpha} \frac{e^{(k-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^k} \right) \left(\frac{(J-k)!}{(J-k)^\alpha} \frac{e^{((J-k)-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^{J-k}} \right) \\ & + \sum_E \sum_{k=\lfloor \frac{J}{2} \rfloor+1}^J \binom{J}{k} c^2 \left(\frac{(k-1)!}{(k-1)^\alpha} \frac{e^{(k-3)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^{k-1}} \right) \left(\frac{(J+1-k)!}{(J+1-k)^\alpha} \frac{e^{((J+1-k)-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^{J+1-k}} \right) \\ & \leq c \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right)\end{aligned} \quad (14.147)$$

$$\begin{aligned}
& + 2 \sum_E \sum_{k=0}^{\lfloor \frac{J}{2} \rfloor} \binom{J}{k} c^2 \left(\frac{k!}{k^\alpha} \frac{e^{(k-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^k} \right) \left(\frac{(J-k)!}{(J-k)^\alpha} \frac{e^{((J-k)-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^{J-k}} \right) \\
& \leq c \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right) \\
& + \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right) \left[\left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right)^{-1} \right. \\
& \quad \left. \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \left(2^2 c^2 e^{-2(\delta+\Gamma(\lambda))} \right) \sum_{k=0}^{\lfloor \frac{J}{2} \rfloor} \binom{J}{k} \left(\frac{k!}{k^\alpha} \right) \left(\frac{(J-k)!}{(J-k)^\alpha} \right) \right] \\
& \leq c \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right) \\
& + \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right) \left[\left(2^2 c^2 e^{-2(\delta+\Gamma(\lambda))} \right) \sum_{k=0}^{\lfloor \frac{J}{2} \rfloor} \left(\frac{1}{k^\alpha} \right) \left(\frac{J^\alpha}{(J-k)^\alpha} \right) \right] \\
& \leq c \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right) + \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right) \left[\left(2^{2+\alpha} c^2 e^{-2(\delta+\Gamma(\lambda))} \right) \sum_{k=0}^{\lfloor \frac{J}{2} \rfloor} \frac{1}{k^\alpha} \right] \\
& \leq \left(c + \frac{c}{2} \right) \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right), \tag{14.148}
\end{aligned}$$

provided that δ is sufficiently large so that,

$$\left[\left(2^{2+\alpha} c e^{-2(\delta+\Gamma(\lambda))} \right) \sum_{k=0}^{\lfloor \frac{J}{2} \rfloor} \frac{1}{k^\alpha} \right] \leq \frac{1}{2}. \tag{14.149}$$

Choosing suitably the constants c , we have proved the inequality

$$|r^{((J+1)-\frac{2}{p})} \mathbf{g}(e_E, \mathcal{L}_O^J \nabla e_A)|_{p,S} \leq c \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right), \tag{14.150}$$

Once we have the estimates 14.150 we obtain the estimates for $|r^{((J+1)-\frac{2}{p})} \mathbf{g}(e_E, \mathcal{L}_O^{J+1} e_A)|_{p,S}$ applying corollary 9.5.

14.11 Some extra remarks on the canonical foliation

To see in a more clear way why the canonical foliation has to be used we have first to discuss why technically this problem arises and subsequently look for

a more “deep” justification. Let us start with the technical aspects. As discussed before to get the correct decays we need to control some connection coefficient norms integrating from above. This requires, specifically for $\tilde{\mu}$ and $\tilde{\omega}$, the control of these quantities on \underline{C}_* . To do it we have integrate them along \underline{C}_* , therefore along an incoming direction which is not the natural one for these quantities associated to non underlined connection coefficients. In fact if one tries to do it along \underline{C}_* looking at its transport equation in a generic foliation it follows immediately that a loss of derivatives appears. In fact as $\tilde{\mu} = -\mathfrak{d}\mathfrak{v}\eta + \frac{1}{2}(\chi\underline{\chi} - \overline{\chi}\overline{\underline{\chi}}) - (\rho - \bar{\rho})$ it seems evident that computing $\mathfrak{D}_3\tilde{\mu}$ on the right hand side we obtain terms of the order of angular derivatives of a Riemann component, this happens both \mathfrak{D}_3 deriving $-\mathfrak{d}\mathfrak{v}\eta$ and ρ ; on the other side $\tilde{\mu}$ is of order of a Riemann component (one derivative of connection coefficients) and therefore a loss of derivative is present. The choice of the canonical foliation solves this problem and, moreover, guarantees the better decay for the connection components along the null outgoing directions. Moreover no loss of derivatives is present imposing the initial data on \underline{C}_* for $\tilde{\omega}$. Once we realize that to estimate $\tilde{\mu}$ we have to integrate from above and, therefore, for getting the right estimates for $\tilde{\mu}|_{\underline{C}_*}$ we need to choose an appropriate (canonical) foliation, it turns out that we have to use this foliation for the transport of all the not underlined quantities which, therefore, all of them, have to be integrated from above. This implies that the chosen foliation, that is Ω , has to be such that even $\text{tr}\chi$ and its derivatives (or Ψ which is basically the same thing) when transported along \underline{C}_* must avoid any loss of derivatives. This is possible to obtain with the right choice of $\Omega|_{\underline{C}_*}$ as we discuss in more detail in the following. Let us start writing the transport equation for $\tilde{\mu}$ in a generic foliation. In that case

$$\tilde{\mu} = -\mathfrak{d}\mathfrak{v}\eta + \frac{1}{2}(\chi\underline{\chi} - \overline{\chi}\overline{\underline{\chi}}) - (\rho - \bar{\rho}) .$$

Therefore

$$\begin{aligned} \mathfrak{D}_3\mathfrak{d}\mathfrak{v}\eta &= \mathfrak{d}\mathfrak{v}\mathfrak{D}_3(-\underline{\eta} + 2\underline{\nabla}\log\Omega) + [\mathfrak{D}_3, \mathfrak{d}\mathfrak{v}]\eta \\ &= \mathfrak{d}\mathfrak{v} [\underline{\chi} \cdot \underline{\eta} - \underline{\chi} \cdot \eta - \underline{\beta}] + 2\mathfrak{d}\mathfrak{v}\mathfrak{D}_3\underline{\nabla}\log\Omega + [\mathfrak{D}_3, \mathfrak{d}\mathfrak{v}]\eta \\ &= \mathfrak{d}\mathfrak{v} [\underline{\chi} \cdot \underline{\eta} - \underline{\chi} \cdot \eta - \underline{\beta}] + 2\underline{\Delta}\mathfrak{D}_3\log\Omega + 2\mathfrak{d}\mathfrak{v}[\mathfrak{D}_3, \underline{\nabla}]\log\Omega + [\mathfrak{D}_3, \mathfrak{d}\mathfrak{v}]\eta \\ &= -\mathfrak{d}\mathfrak{v}\underline{\beta} + \mathfrak{d}\mathfrak{v} [\underline{\chi} \cdot \underline{\eta} - \underline{\chi} \cdot \eta] + 2\underline{\Delta}\mathfrak{D}_3\log\Omega + 2\mathfrak{d}\mathfrak{v}[\mathfrak{D}_3, \underline{\nabla}]\log\Omega + [\mathfrak{D}_3, \mathfrak{d}\mathfrak{v}]\eta \\ &= -\mathfrak{d}\mathfrak{v}\underline{\beta} - 4\underline{\Delta}\underline{\omega} + \{\mathfrak{d}\mathfrak{v} [\underline{\chi} \cdot \underline{\eta} - \underline{\chi} \cdot \eta] + 2\mathfrak{d}\mathfrak{v}[\mathfrak{D}_3, \underline{\nabla}]\log\Omega + [\mathfrak{D}_3, \mathfrak{d}\mathfrak{v}]\eta\} \end{aligned}$$

which we can rewrite as

$$\mathfrak{D}_3\mathfrak{d}\mathfrak{v}\eta = -\mathfrak{d}\mathfrak{v}\underline{\beta} - 4\underline{\Delta}\underline{\omega} + [\tilde{\mu}.o.t] . \quad (14.151)$$

Recall that $\tilde{\mu}$ is at the level of the first derivative of the connection coefficients, that is at the “Riemann level”, moreover

$$\frac{1}{2}\mathfrak{D}_3(\chi\underline{\chi} - \overline{\chi}\overline{\underline{\chi}}) = [\tilde{\mu}.o.t] \quad (14.152)$$

where with $[\tilde{\mu}.o.t]$ we indicate terms at the order of $\tilde{\mu}$ which, therefore do not imply any loss of derivatives in the transport equation. Finally

$$\mathfrak{D}_3(\rho - \bar{\rho}) = \mathfrak{D}_3\rho + [\tilde{\mu}.o.t] = -\mathfrak{d}\mathfrak{iv}\underline{\beta} + [\tilde{\mu}.o.t] \quad (14.153)$$

and finally

$$\begin{aligned} \mathfrak{D}_3\tilde{\mu} &= -\mathfrak{D}_3\mathfrak{d}\mathfrak{iv}\eta + \frac{1}{2}\mathfrak{D}_3(\chi\underline{\chi} - \overline{\chi\underline{\chi}}) - \mathfrak{D}_3(\rho - \bar{\rho}) \\ &= \mathfrak{d}\mathfrak{iv}\underline{\beta} + 4\underline{\Delta}\underline{\omega} + \mathfrak{d}\mathfrak{iv}\underline{\beta} + [\tilde{\mu}.o.t] = 4\underline{\Delta}\underline{\omega} + 2\mathfrak{d}\mathfrak{iv}\underline{\beta} + [\tilde{\mu}.o.t] \\ &= -2\underline{\Delta}\mathfrak{D}_3\log\Omega + 2\mathfrak{d}\mathfrak{iv}\underline{\beta} + [\tilde{\mu}.o.t] = -2\mathfrak{D}_3\underline{\Delta}\log\Omega + 2\mathfrak{d}\mathfrak{iv}\underline{\beta} + [\tilde{\mu}.o.t] . \end{aligned}$$

Therefore to avoid the loss of derivatives in the transport equation for $\tilde{\mu}|_{\underline{C}_*}$ we have to choose $\Omega|_{\underline{C}_*}$ in such a way that $-2\mathfrak{D}_3\underline{\Delta}\log\Omega + 2\mathfrak{d}\mathfrak{iv}\underline{\beta}$ is of the same order as $\tilde{\mu}$. This still leaves many possibilities for the choice of Ω , but, as we said, once we choose $\Omega|_{\underline{C}_*}$ and we define the outgoing cones of the foliation we need to transport all the non underlined quantities starting from above, therefore we do it also for $\text{tr}\chi$ and its angular derivatives or for ψ even if in principle they have the right decay even if integrating them from below. This imposes a choice of $\Omega|_{\underline{C}_*}$ such that the transport equation of $\text{tr}\chi$ and its derivatives along \underline{C}_* does not have any loss of derivatives and this defines $\Omega|_{\underline{C}_*}$ as done in [Kl-Ni2], a choice which simultaneously control also the evolution equation for $\tilde{\mu}|_{\underline{C}_*}$. Let us see it in a constructive way: the equation along the incoming cones for $\text{tr}\chi$ is

$$\mathfrak{D}_3\text{tr}\chi + \text{tr}\underline{\chi}\text{tr}\chi - 2\underline{\omega}\text{tr}\chi - 2\mathfrak{d}\mathfrak{iv}\eta - 2|\eta|^2 + 2\mathbf{K} = 0 . \quad (14.154)$$

which we rewrite using the relation

$$\mathbf{K} + \frac{1}{4}\text{tr}\chi\text{tr}\underline{\chi} = \frac{1}{2}\hat{\chi} \cdot \hat{\underline{\chi}} - \rho , \quad (14.155)$$

as

$$\mathfrak{D}_3\text{tr}\chi + \frac{1}{2}\text{tr}\underline{\chi}\text{tr}\chi - 2\underline{\omega}\text{tr}\chi - 2|\eta|^2 - 2\mathfrak{d}\mathfrak{iv}\eta + \hat{\chi} \cdot \hat{\underline{\chi}} - 2\rho = 0 . \quad (14.156)$$

Recalling that

$$2\tilde{\mu} := -2\mathfrak{d}\mathfrak{iv}\eta + (\hat{\chi}\hat{\underline{\chi}} - \overline{\hat{\chi}\hat{\underline{\chi}}}) - 2(\rho - \bar{\rho}) + \frac{1}{2}(\text{tr}\chi\text{tr}\underline{\chi} - \overline{\text{tr}\chi\text{tr}\underline{\chi}}) \quad (14.157)$$

we rewrite the transport equation for $\text{tr}\chi$ as

$$\begin{aligned} \mathfrak{D}_3\text{tr}\chi + \frac{1}{2}\text{tr}\underline{\chi}\text{tr}\chi - 2\underline{\omega}\text{tr}\chi - 2|\eta|^2 + \hat{\chi}\hat{\underline{\chi}} - 2\rho + 2\tilde{\mu} - (\hat{\chi}\hat{\underline{\chi}} - \overline{\hat{\chi}\hat{\underline{\chi}}}) + 2(\rho - \bar{\rho}) - \frac{1}{2}(\text{tr}\chi\text{tr}\underline{\chi} - \overline{\text{tr}\chi\text{tr}\underline{\chi}}) \\ = \mathfrak{D}_3\text{tr}\chi + \frac{1}{2}\overline{\text{tr}\chi\text{tr}\underline{\chi}} - 2\underline{\omega}\text{tr}\chi - 2|\eta|^2 + \overline{\hat{\chi}\hat{\underline{\chi}}} - 2\bar{\rho} + 2\tilde{\mu} = 0 . \end{aligned} \quad (14.158)$$

Therefore the equation which $\text{tr}\chi$ satisfies, in a general foliation, is

$$\mathfrak{D}_3\text{tr}\chi + \frac{1}{2}\overline{\text{tr}\chi\text{tr}\underline{\chi}} - 2\underline{\omega}\text{tr}\chi - 2|\eta|^2 + \overline{\hat{\chi}\hat{\underline{\chi}}} - 2\bar{\rho} + 2\tilde{\mu} = 0 , \quad (14.159)$$

and if we want that there is no loss of derivatives we need that the order of $\tilde{\mu}$ be that of a connection coefficient. Therefore looking at the expression for $\tilde{\mu}$,

$$\tilde{\mu} = -\mathfrak{d}\mathfrak{i}\mathfrak{v}\eta + \frac{1}{2}(\chi\underline{\chi} - \overline{\chi\underline{\chi}}) - (\rho - \overline{\rho}) ,$$

we are forced to impose that

$$-\mathfrak{d}\mathfrak{i}\mathfrak{v}\eta - \rho = 0 + O(\overline{\rho} + [\text{conn.coef.order}]) \quad (14.160)$$

implying for $\text{tr}\chi$ the transport equation

$$\begin{aligned} & \mathfrak{D}_3 \text{tr}\chi + \frac{1}{2} \overline{\text{tr}\chi \text{tr}\underline{\chi}} - 2\underline{\omega} \text{tr}\chi - 2|\eta|^2 + \overline{\hat{\chi}\hat{\chi}} - 2\overline{\rho} + (\chi\underline{\chi} - \overline{\chi\underline{\chi}}) + O(\overline{\rho} + [\text{conn.coef.order}]) \\ &= \mathfrak{D}_3 \text{tr}\chi + \frac{1}{2} \text{tr}\underline{\chi} \text{tr}\chi - 2\underline{\omega} \text{tr}\chi - 2|\eta|^2 + \hat{\chi}\hat{\chi} + O(\overline{\rho} + [\text{conn.coef.order}]) = 0 . \end{aligned} \quad (14.161)$$

If we assume in 14.160, $O(\overline{\rho} + [\text{conn.coef.order}]) = 0$ then the equation satisfied on $\underline{\mathcal{C}}_*$ by $\text{tr}\chi$ is

$$\mathfrak{D}_3 \text{tr}\chi + \frac{1}{2} \text{tr}\underline{\chi} \text{tr}\chi - 2\underline{\omega} \text{tr}\chi - 2|\eta|^2 + \hat{\chi}\hat{\chi} = 0 \quad (14.162)$$

and Ω has to satisfy

$$\mathbb{A}\Omega = -\mathfrak{d}\mathfrak{i}\mathfrak{v}\zeta - \rho . \quad (14.163)$$

If we assume in 14.160, $O(\overline{\rho} + [\text{conn.coef.order}]) = -\overline{\rho}$ then the equation satisfied on $\underline{\mathcal{C}}_*$ by $\text{tr}\chi$ is

$$\mathfrak{D}_3 \text{tr}\chi + \frac{1}{2} \text{tr}\underline{\chi} \text{tr}\chi - 2\underline{\omega} \text{tr}\chi - 2|\eta|^2 + \hat{\chi}\hat{\chi} - 2\overline{\rho} = 0 \quad (14.164)$$

and Ω has to satisfy

$$\mathbb{A}\Omega = -\mathfrak{d}\mathfrak{i}\mathfrak{v}\zeta - (\rho - \overline{\rho}) . \quad (14.165)$$

Remarks:

a) Observe that here what we are really interested are the estimates of inductive estimates of the angular derivatives, the estimates for the first 3 derivaties are already done in [Kl-Ni2]. For the higher derivative estimates of $\text{tr}\chi$ we use equations 14.162 or 14.164 and for the angular derivatives of η and of $\hat{\chi}$ and $\underline{\hat{\chi}}$ in 14.164 we use the Hodge systems for $\hat{\chi}$ and $\underline{\hat{\chi}}$ and for η the Hodge system valid on $\underline{\mathcal{C}}_*$ 9.45

$$\begin{aligned} \mathfrak{d}\mathfrak{i}\mathfrak{v}\eta &= -\tilde{\mu}|_{\underline{\mathcal{C}}_*} + \frac{1}{2}(\chi\underline{\chi} - \overline{\chi\underline{\chi}}) - (\rho - \overline{\rho}) \\ \text{curl}\eta &= \sigma - \frac{1}{2}\hat{\chi} \wedge \hat{\chi} . \end{aligned}$$

b) From the previous considerations it seems that there is still a certain freedom in specifying the $\Omega|_{\underline{\mathcal{C}}_*}$ which defines the canonical foliation. In fact in principle it seems that up to now requiring that $\Omega|_{\underline{\mathcal{C}}_*}$ satisfies the elliptic equations 14.160 or 14.163 or even 14.165 is at this level completely equivalent. A more stringent condition can be required once we look at the asymptotic behaviours of the Riemann tensor components.

15 Appendix to Section 10

15.1 The definition and the control of the \mathcal{Q} norms

We discuss first of all which are the norms we need to control for $J > 3$; let us recall, first of all, their definitions given in Section 4:

We define, for $J \geq 1$,¹¹⁵

$$\begin{aligned}\mathcal{Q}^{(J-1)}(\lambda, \nu) &= \mathcal{Q}_1^{(J-1)}(\lambda, \nu) + \mathcal{Q}_2^{(J-1)}(\lambda, \nu) \\ \underline{\mathcal{Q}}^{(J-1)}(\lambda, \nu) &= \underline{\mathcal{Q}}_1^{(J-1)}(\lambda, \nu) + \underline{\mathcal{Q}}_2^{(J-1)}(\lambda, \nu)\end{aligned}\quad (15.1)$$

Remark:

Recall that $Q(\hat{\mathcal{L}}_O W) = \sum_{i=1}^3 Q(\hat{\mathcal{L}}_{O_{(i)}} W)$.

Therefore with our definitions

$$\begin{aligned}\mathcal{Q}_1^{(J-1)}(\lambda, \nu) &\equiv \sum_{i_1, i_2, \dots, i_{J-1}}^{\{1,3\}} \int_{C(\lambda) \cap V(\lambda, \nu)} Q(\hat{\mathcal{L}}_T(\hat{\mathcal{L}}_{O_{(i_1)}} \hat{\mathcal{L}}_{O_{(i_2)}} \cdots \hat{\mathcal{L}}_{O_{(i_{J-1})}} W))(\bar{K}, \bar{K}, \bar{K}, e_4) \\ &+ \sum_{i_0, i_1, i_2, \dots, i_{J-1}}^{\{1,3\}} \int_{C(\lambda) \cap V(\lambda, \nu)} Q(\hat{\mathcal{L}}_{O_{(i_0)}}(\hat{\mathcal{L}}_{O_{(i_1)}} \hat{\mathcal{L}}_{O_{(i_2)}} \cdots \hat{\mathcal{L}}_{O_{(i_{J-1})}} W))(\bar{K}, \bar{K}, T, e_4) .\end{aligned}$$

We write in a compact way, but recalling the previous remark,

$$\begin{aligned}\mathcal{Q}_1^{(J-1)}(\lambda, \nu) &\equiv \int_{C(\lambda) \cap V(\lambda, \nu)} Q(\hat{\mathcal{L}}_T(\hat{\mathcal{L}}_O^{J-1} W))(\bar{K}, \bar{K}, \bar{K}, e_4) \\ &+ \int_{C(\lambda) \cap V(\lambda, \nu)} Q(\hat{\mathcal{L}}_O(\hat{\mathcal{L}}_O^{J-1} W))(\bar{K}, \bar{K}, T, e_4) \\ \mathcal{Q}_2^{(J-1)}(\lambda, \nu) &\equiv \int_{C(\lambda) \cap V(\lambda, \nu)} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T(\hat{\mathcal{L}}_O^{J-1} W))(\bar{K}, \bar{K}, \bar{K}, e_4) \\ &+ \int_{C(\lambda) \cap V(\lambda, \nu)} Q(\hat{\mathcal{L}}_O^2(\hat{\mathcal{L}}_O^{J-1} W))(\bar{K}, \bar{K}, T, e_4) \\ &+ \int_{C(\lambda) \cap V(\lambda, \nu)} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T(\hat{\mathcal{L}}_O^{J-1} W))(\bar{K}, \bar{K}, \bar{K}, e_4)\end{aligned}\quad (15.2)$$

$$\begin{aligned}\underline{\mathcal{Q}}_1^{(J-1)}(\lambda, \nu) &\equiv \sup_{V(\lambda, \nu) \cap C_0} |r^3 \bar{\rho}(\hat{\mathcal{L}}_O^{J-1} W)|^2 + \int_{\underline{C}(\nu) \cap V(\lambda, \nu)} Q(\hat{\mathcal{L}}_T(\hat{\mathcal{L}}_O^{J-1} W))(\bar{K}, \bar{K}, \bar{K}, e_3) \\ &+ \int_{\underline{C}(\nu) \cap V(\lambda, \nu)} Q(\hat{\mathcal{L}}_O(\hat{\mathcal{L}}_O^{J-1} W))(\bar{K}, \bar{K}, T, e_3) \\ \underline{\mathcal{Q}}_2^{(J-1)}(\lambda, \nu) &\equiv \int_{\underline{C}(\nu) \cap V(\lambda, \nu)} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T(\hat{\mathcal{L}}_O^{J-1} W))(\bar{K}, \bar{K}, \bar{K}, e_3)\end{aligned}$$

¹¹⁵For the definitions of the vector fields O, K, T and the modified Lie derivatives $\hat{\mathcal{L}}$. see [Kl-Ni2] Chapter 3.

$$\begin{aligned}
& + \int_{\underline{C}(\nu) \cap V(\lambda, \nu)} Q(\hat{\mathcal{L}}_O^2(\hat{\mathcal{L}}_O^{J-1}W))(\bar{K}, \bar{K}, T, e_3) \\
& + \int_{\underline{C}(\nu) \cap V(\lambda, \nu)} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T(\hat{\mathcal{L}}_O^{J-1}W))(\bar{K}, \bar{K}, \bar{K}, e_3)
\end{aligned} \tag{15.3}$$

where $V(\lambda, \nu) = J^{(+)}(S(\lambda_0, \nu_0) \cap J^{(-)}(S(\lambda, \nu)))$.

Proof of lemma 10.2: Let us consider a generic value $H \leq J - 1$, with $\tilde{W}^{(H)} = \hat{\mathcal{L}}_O^H W$ we have,

$$Q(\tilde{W}^{(H)})(\bar{K}, \bar{K}, T, e_4) = \frac{1}{4} \tau_+^4 \left(|\alpha(\tilde{W}^{(H)})|^2 \right) + \dots \tag{15.4}$$

and the analogous expressions for $Q(\tilde{W}^{(H)})(\bar{K}, \bar{K}, \bar{K}, e_4)$, $Q(\tilde{W}^{(H)})(\bar{K}, \bar{K}, T, e_3)$, $Q(\tilde{W}^{(H)})(\bar{K}, \bar{K}, T, e_4)$, Recall that

$$\begin{aligned}
|rF|_{4, S(\lambda, \nu)} & \leq |rF|_{4, S(\lambda, \nu_0)} + c \left(+ \int_{C(\lambda) \cap V(\lambda, \nu)} [|F|^2 + |r\nabla F|^2 + |r\mathcal{D}_4 F|^2] \right)^{\frac{1}{2}} \\
|r^2 \nabla F|_{4, S(\lambda, \nu)} & \leq |r^2 \nabla F|_{4, S(\lambda, \nu_0)} + c \left(+ \int_{C(\lambda) \cap V(\lambda, \nu)} [|r\nabla F|^2 + |r^2 \nabla^2 F|^2 + |r^2 \nabla \mathcal{D}_4 F|^2] \right)^{\frac{1}{2}}.
\end{aligned} \tag{15.5}$$

Therefore

$$|r^{\frac{9}{2}-\frac{2}{4}} \nabla \alpha|_{4, S(\lambda, \nu)} \leq |r^{\frac{9}{2}-\frac{2}{4}} \nabla \alpha|_{4, S(\lambda, \nu_0)} + c \left(+ \int_{C(\lambda) \cap V(\lambda, \nu)} [r^3 |\nabla \alpha|^2 + |r^4 \nabla^2 \alpha|^2 + |r^4 \nabla \mathcal{D}_4 \alpha|^2] \right)^{\frac{1}{2}}$$

and, omitting the constants in front of the integrals, using Lemma 5.1.1 of [Kl-Ni2] and omitting correction terms,¹¹⁶

$$\begin{aligned}
& \int_{C(\lambda) \cap V(\lambda, \nu)} |r^3 \nabla \alpha|^2 = \int_{\nu_0}^{\nu} d\nu' \int_{S(\lambda, \nu')} r^6 |\nabla \alpha|^2 = \int_{\nu_0}^{\nu} d\nu' r^4 \int_{S(\lambda, \nu')} |r \nabla \alpha|^2 \\
& \leq \int_{\nu_0}^{\nu} d\nu' r^4 \int_{S(\lambda, \nu')} |\hat{\mathcal{L}}_O \alpha|^2, \\
& \int_{C(\lambda) \cap V(\lambda, \nu)} |r^4 \nabla^2 \alpha|^2 = \int_{\nu_0}^{\nu} d\nu' \int_{S(\lambda, \nu')} r^8 |\nabla^2 \alpha|^2 = \int_{\nu_0}^{\nu} d\nu' r^4 \int_{S(\lambda, \nu')} |r^2 \nabla^2 \alpha|^2 \\
& \leq \int_{\nu_0}^{\nu} d\nu' r^4 \int_{S(\lambda, \nu')} |\hat{\mathcal{L}}_O^2 \alpha|^2.
\end{aligned}$$

In an analogous way we can show that

$$\int_{C(\lambda) \cap V(\lambda, \nu)} |r^4 \nabla \mathcal{D}_4 \alpha|^2 \leq \int_{\nu_0}^{\nu} d\nu' r^6 \int_{S(\lambda, \nu')} |\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T \alpha|^2.$$

¹¹⁶ Arising from passing from \mathcal{L}_O to $\hat{\mathcal{L}}_O$ and from the commutation terms arising interchanging \mathcal{L}_O and $r\nabla$.

Therefore

$$\begin{aligned} |r^{\frac{9}{2}-\frac{2}{p}}\nabla\alpha|_{p=4,S(\lambda,\nu)} &\leq |r^{\frac{9}{2}-\frac{2}{p}}\nabla\alpha|_{p=4,S(\lambda,\nu_0)} \\ &+ \int_{\nu_0}^{\nu} d\nu' r^4 \int_{S(\lambda,\nu')} |\hat{\mathcal{L}}_O\alpha|^2 + \int_{\nu_0}^{\nu} d\nu' r^4 \int_{S(\lambda,\nu')} |\hat{\mathcal{L}}_O^2\alpha|^2 + \int_{\nu_0}^{\nu} d\nu' r^6 \int_{S(\lambda,\nu')} |\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T\alpha|^2. \end{aligned} \quad (15.6)$$

It follows,

$$\begin{aligned} |r^{\frac{9}{2}-\frac{2}{p}}\nabla\alpha(\tilde{W}^{(H)})|_{p=4,S(\lambda,\nu)} &= \left(\int_{S(\lambda,\nu)} d\mu \left(\left| r^4 \nabla\alpha(\tilde{W}^{(H)}) \right|^2 \right)^2 \right)^{\frac{1}{4}} \\ &= \left(\int_{S(\lambda,\nu)} d\mu |r^4 \nabla\alpha(\tilde{W}^{(H)})|^4 \right)^{\frac{1}{4}} \end{aligned} \quad (15.7)$$

The relation becomes, considering $\tilde{W}^{(H)} = \hat{\mathcal{L}}_O^H W$,

$$\begin{aligned} |r^{\frac{9}{2}-\frac{2}{p}}\nabla\alpha(\tilde{W}^{(H)})|_{p=4,S(\lambda,\nu)} &\leq |r^{\frac{9}{2}-\frac{2}{p}}\nabla\alpha(\tilde{W}^{(H)})|_{p=4,S(\lambda,\nu_0)} + \int_{\nu_0}^{\nu} d\nu' r^4 \int_{S(\lambda,\nu')} |\hat{\mathcal{L}}_O\alpha(\tilde{W}^{(H)})|^2 \\ &+ \int_{\nu_0}^{\nu} d\nu' r^4 \int_{S(\lambda,\nu')} |\hat{\mathcal{L}}_O^2\alpha(\tilde{W}^{(H)})|^2 + \int_{\nu_0}^{\nu} d\nu' r^6 \int_{S(\lambda,\nu')} |\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T\alpha(\tilde{W}^{(H)})|^2 \end{aligned} \quad (15.8)$$

and, more explicitly,

$$\begin{aligned} \int_{\nu_0}^{\nu} d\nu' r^4 \int_{S(\lambda,\nu')} |\hat{\mathcal{L}}_O\alpha(\tilde{W}^{(H)})|^2 &\leq \int_{\nu_0}^{\nu} d\nu' r^4 |\alpha(\hat{\mathcal{L}}_O\tilde{W}^{(H)})|^2 \\ &\leq \int_{\nu_0}^{\nu} d\nu' Q(\hat{\mathcal{L}}_O\tilde{W}^{(H)})(\bar{K}, \bar{K}, T, e_4). \end{aligned}$$

Therefore in conclusion we have

$$\begin{aligned} |r^{\frac{9}{2}-\frac{2}{p}}\nabla\alpha(\tilde{W}^{(H)})|_{p=4,S(\lambda,\nu)} &\leq |r^{\frac{9}{2}-\frac{2}{p}}\nabla\alpha(\tilde{W}^{(H)})|_{p=4,S(\lambda,\nu_0)} \\ &+ \int_{\nu_0}^{\nu} d\nu' Q(\hat{\mathcal{L}}_O\tilde{W}^{(H)})(\bar{K}, \bar{K}, T, e_4) + \int_{\nu_0}^{\nu} d\nu' Q(\hat{\mathcal{L}}_O^2\tilde{W}^{(H)})(\bar{K}, \bar{K}, T, e_4) \\ &+ \int_{\nu_0}^{\nu} d\nu' Q(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T\tilde{W}^{(H)})(\bar{K}, \bar{K}, \bar{K}, e_4). \end{aligned} \quad (15.9)$$

Therefore if we control $\mathcal{Q}_{1,2}^{(J-2)}$ we control the norm $|r^{\frac{9}{2}-\frac{2}{p}}\nabla\alpha(\tilde{W}^{(J-2)})|_{p,S(\lambda,\nu)}$ and with a slight modification it follows that we control $|r^{\frac{7}{2}-\frac{2}{p}+(J-1)}\hat{\mathcal{L}}_O^{J-1}\alpha(W)|_{p,S}$.¹¹⁷

15.2 Proof of Lemma 4.1

To estimate $\nabla_O^{J-1}\Psi$ we proceed in the following way, we write symbolically,

$$\nabla_O^{J-1}\Psi = O\nabla_O\nabla_O\cdots O\nabla_O\Psi.$$

¹¹⁷ On the other side if we consider only $\mathcal{Q}_1^{(J-1)}$ we can also control directly $|r^{\frac{9}{2}-\frac{2}{p}}\alpha(\tilde{W}^{(J-1)})|_{p,S(\lambda,\nu)}$.

We consider the O 's as endowed with a given name, numbered, each one, with the exception of the first, defining a "slot" namely the position at its left, therefore there are $J-2$ slots associated to the $(J-2)$ O 's we are considering; we add an extra slot which corresponds to the position immediately at the left of Ψ . Therefore we have $(J-1)$ slots and we imagine to distribute in all the possible ways the $(J-1)$ ∇ in these slots. Clearly in this way we are counting more than the possible terms we obtain writing explicitly $\nabla_O^{J-1}\Psi$ as each ∇ cannot operate on the O 's at its left. Assuming the described distribution of the ∇ 's we have (the equality is formal and it has to be interpreted as an upper bounds when we consider the norms),

$$\nabla_O^{J-1}\Psi = "O \left\{ \sum_{\substack{s=1 \\ \gamma_1, \gamma_2, \dots, \gamma_{J-2} \geq 1}}^{J-1} \frac{(J-1)!}{\gamma_1! \gamma_2! \dots \gamma_{J-2}! \gamma_{J-1}!} \nabla^{\gamma_1} O \nabla^{\gamma_2} O \dots \nabla^{\gamma_{J-2}} O \nabla^{\gamma_{J-1}} \Psi \right\} . \quad (15.10)$$

We denote $\gamma_{J-1} = q$ and rewrite the previous expression as

$$\nabla_O^{J-1}\Psi = "O \left\{ \sum_{q=1}^{J-1} \binom{J-1}{q} \nabla^q \Psi \sum_{\gamma_1, \gamma_2, \dots, \gamma_{J-2}}^{\sum_{s=1}^{J-2} \gamma_s = (J-1-q)} \frac{(J-1-q)!}{\gamma_1! \gamma_2! \dots \gamma_{J-2}!} \nabla^{\gamma_1} O \nabla^{\gamma_2} O \dots \nabla^{\gamma_{J-2}} O \right\} . \quad (15.11)$$

Denoting $k = J-1-q$ we rewrite the previous expression as

$$\begin{aligned} \nabla_O^{J-1}\Psi &= "O \left\{ \sum_{k=0}^{J-2} \binom{J-1}{k} \nabla^{J-1-k} \Psi \sum_{\gamma_1, \gamma_2, \dots, \gamma_{J-2}}^{\sum_{s=1}^{J-2} \gamma_s = k} \frac{k!}{\gamma_1! \gamma_2! \dots \gamma_{J-2}!} \nabla^{\gamma_1} O \nabla^{\gamma_2} O \dots \nabla^{\gamma_{J-2}} O \right\} \\ &= O^{J-1} \nabla^{J-1} \Psi + O \left\{ \sum_{k=1}^{J-2} \binom{J-1}{k} \nabla^{J-1-k} \Psi \sum_{\gamma_1, \gamma_2, \dots, \gamma_{J-2}}^{\sum_{s=1}^{J-2} \gamma_s = k} \frac{k!}{\gamma_1! \gamma_2! \dots \gamma_{J-2}!} \nabla^{\gamma_1} O \nabla^{\gamma_2} O \dots \nabla^{\gamma_{J-2}} O \right\} . \end{aligned}$$

Going to the norms the estimate becomes

$$\begin{aligned} |\nabla_O^{J-1}\Psi(R)|_{p,S} &\leq |O|_{\infty}^{J-1} |\nabla^{J-1}\Psi(R)|_{p,S} \\ &+ |O|_{\infty} \left\{ \sum_{k=1}^{J-2} \binom{J-1}{k} |\nabla^{J-1-k}\Psi(R)|_{p,S} \sum_{\gamma_1, \gamma_2, \dots, \gamma_{J-2}}^{\sum_{s=1}^{J-2} \gamma_s = k} \frac{k!}{\gamma_1! \gamma_2! \dots \gamma_{J-2}!} \left[|\nabla^{\gamma_1} O|_{\infty, S} |\nabla^{\gamma_2} O|_{\infty, S} \dots |\nabla^{\gamma_{J-2}} O|_{\infty} \right] \right\} . \end{aligned} \quad (15.12)$$

Recalling theorem 9.6 we have the estimates

$$|\nabla^{\gamma} O|_{\infty, S} \leq C_{0,0} \frac{(\gamma+1)!}{(\gamma+1)^{\alpha}} \frac{e^{(\gamma-1)\delta_0} e^{(\gamma-1)\Gamma_0(\lambda)}}{\rho_0^{\gamma+1}} , \gamma > 0 ; |O|_{\infty, S} \leq c .$$

From these estimates it follows that

$$\begin{aligned} &\delta \left(\sum_{s=1}^{J-2} \gamma_s = k \right) \left[|\nabla^{\gamma_1} O|_{\infty, S} |\nabla^{\gamma_2} O|_{\infty, S} \dots |\nabla^{\gamma_{J-2}} O|_{\infty} \right] \\ &\leq |O|_{\infty, S}^{(J-1-k)} \cdot \frac{e^{k(\delta_0 + \Gamma_0(\lambda))}}{\rho_0^k} \frac{(\gamma_1+1)! (\gamma_2+2)! \dots (\gamma_{J-2}+1)! e^{-(\delta_0 + \Gamma_0(\lambda))(\epsilon(\gamma_1) + \epsilon(\gamma_2) + \dots + \epsilon(\gamma_{J-2}))}}{((1 - \epsilon(\gamma_1)) + \gamma_1^{\alpha})((1 - \epsilon(\gamma_2)) + \gamma_2^{\alpha}) \dots ((1 - \epsilon(\gamma_{J-2})) + \gamma_{J-2}^{\alpha})} \end{aligned} \quad (15.13)$$

and we can rewrite the previous estimate, recalling that $|O|_{\infty,S} \geq 1$, as

$$|\nabla_O^{J-1}\Psi(R)|_{p,S} \leq |O|_{\infty,S}^{J-1} |\nabla^{J-1}\Psi(R)|_{p,S} + |O|_{\infty,S}^{J-2} \left\{ \sum_{k=1}^{J-2} \binom{J-1}{k} |\nabla^{J-1-k}\Psi(R)|_{p,S} \frac{e^{k(\delta_0+\Gamma_0(\lambda))}}{\rho_0^k} k! \cdot \right. \\ \left. \cdot \left[\sum_{\gamma_1, \gamma_2, \dots, \gamma_{J-2}}^{\sum_{s=1}^{J-2} \gamma_s = k} \frac{C_{0,0}^{(\epsilon(\gamma_1)+\epsilon(\gamma_2)+\dots+\epsilon(\gamma_{J-2}))} e^{-(\delta_0+\Gamma_0(\lambda))(\epsilon(\gamma_1)+\epsilon(\gamma_2)+\dots+\epsilon(\gamma_{J-2}))}}{((1-\epsilon(\gamma_1)) + \gamma_1^{\alpha-1})(1-\epsilon(\gamma_2)) + \gamma_2^{\alpha-1}) \dots ((1-\epsilon(\gamma_{J-2})) + \gamma_{J-2}^{\alpha-1})} \right] \right\}.$$

Applying the previous estimates for the various terms we have

$$|\nabla_O^{J-1}\Psi(R)|_{p,S} \leq |O|_{\infty,S}^{J-1} \left\{ C_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)}}{\rho_0^J} \right. \\ \left. + \sum_{k=1}^{J-2} \binom{J-1}{k} C_{0,0} \frac{(J-k)!}{(J-k)^\alpha} \frac{e^{((J-k)-2)\delta_0} e^{((J-k)-2)\Gamma_0(\lambda)}}{\rho_0^{J-k}} \frac{e^{k(\delta_0+\Gamma_0(\lambda))}}{\rho_0^k} k! |\mathcal{F}| \right\} \quad (15.14)$$

where

$$|\mathcal{F}| \equiv \left[\sum_{\gamma_1, \gamma_2, \dots, \gamma_{J-2}}^{\sum_{s=1}^{J-2} \gamma_s = k} \frac{C_{0,0}^{(\epsilon(\gamma_1)+\epsilon(\gamma_2)+\dots+\epsilon(\gamma_{J-2}))} e^{-(\delta_0+\Gamma_0(\lambda))(\epsilon(\gamma_1)+\epsilon(\gamma_2)+\dots+\epsilon(\gamma_{J-2}))}}{((1-\epsilon(\gamma_1)) + \gamma_1^{\alpha-1})(1-\epsilon(\gamma_2)) + \gamma_2^{\alpha-1}) \dots ((1-\epsilon(\gamma_{J-2})) + \gamma_{J-2}^{\alpha-1})} \right]. \quad (15.15)$$

Therefore we rewrite 15.108 as

$$|\nabla_O^{J-1}\Psi(R)|_{p,S} \leq |O|_{\infty,S}^{J-1} \left(C_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)}}{\rho_0^J} \right) \left\{ 1 + \sum_{k=1}^{J-2} \binom{J-1}{k} \frac{(J-k)!k!}{J!} \frac{J^\alpha}{(J-k)^\alpha} |\mathcal{F}| \right\} \\ \leq |O|_{\infty,S}^{J-1} \left(C_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)}}{\rho_0^J} \right) \left\{ 1 + \sum_{k=1}^{J-2} \frac{(J-k)}{J} \frac{J^\alpha}{(J-k)^\alpha} |\mathcal{F}| \right\} \\ \leq |O|_{\infty,S}^{J-1} \left(C_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)}}{\rho_0^J} \right) \left\{ 1 + \sum_{k=1}^{J-2} \frac{J^{\alpha-1}}{(J-k)^{\alpha-1}} |\mathcal{F}| \right\}. \quad (15.16)$$

We have to estimate $|\mathcal{F}|$, the simplest and rougher estimate is:

$$|\mathcal{F}| \leq C_{0,0} e^{-(\delta_0+\Gamma_0(\lambda))} C_1^{J-2}, \quad (15.17)$$

a better estimate is obtained writing

$$|\mathcal{F}| = C_{0,0} e^{-(\delta_0+\Gamma_0(\lambda))} \left(\sum_{l=1}^k \binom{J-2}{l} C_{0,0}^{l-1} e^{-(l-1)(\delta_0+\Gamma_0(\lambda))} \left(\sum_{\gamma_1, \gamma_2, \dots, \gamma_l}^{\sum_{s=1}^l \gamma_s = k, \gamma_s \geq 1} \frac{1}{\gamma_1^{\alpha-1} \gamma_2^{\alpha-1} \dots \gamma_l^{\alpha-1}} \right) \right) \\ \leq C_{0,0} e^{-(\delta_0+\Gamma_0(\lambda))} \left(1 + c_1 C_{0,0} e^{-(\delta_0+\Gamma_0(\lambda))} \right)^{J-2} \leq \varepsilon c_{0,0} e^{-(\delta_0+\Gamma_0(\lambda))} \left(1 + c_1 \varepsilon c_{0,0} e^{-(\delta_0+\Gamma_0(\lambda))} \right)^{J-2}. \quad (15.18)$$

Finally

$$|\mathcal{F}| \leq \varepsilon c_{0,0} e^{-(\delta_0+\Gamma_0(\lambda))} (1 + c_2 \varepsilon)^{J-2} \quad (15.19)$$

where

$$\sum_{\substack{s=1 \\ \gamma_1, \gamma_2, \dots, \gamma_l}}^l \frac{1}{\gamma_1^{\alpha-1} \gamma_2^{\alpha-1} \dots \gamma_l^{\alpha-1}} \leq c_1^l \quad , \quad c_2 = c_1 c_{0,0} e^{-(\delta_0 + \Gamma_0(\lambda))} . \quad (15.20)$$

Therefore

$$\left| \left\{ 1 + \sum_{k=1}^{J-2} \frac{J^{\alpha-1}}{(J-k)^{\alpha-1}} |\mathcal{F}| \right\} \right| \leq \left(1 + J^{\alpha-1} \varepsilon c_{0,0} e^{-(\delta_0 + \Gamma_0(\lambda))} (1 + c_2 \varepsilon)^{J-2} \right) \quad (15.21)$$

and finally

$$\begin{aligned} |\nabla_O^{J-1} \Psi(R)|_{p,S} \Big|_{C_0 \cup \underline{C}_0} &\leq \left(C_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)}}{\rho_0^J} \right) \left[|O|_{\infty, S}^{J-1} \left(1 + J^{\alpha-1} \varepsilon c_{0,0} e^{-(\delta_0 + \Gamma_0(\lambda))} (1 + c_2 \varepsilon)^{J-2} \right) \right] \\ &\leq |O|_{\infty}^{J-1} \left(C_{0,0} \frac{J!}{J^\alpha} \frac{e^{(J-2)\delta_0} e^{(J-2)\Gamma_0(\lambda)}}{\rho_{0,1}^J} \right) \end{aligned} \quad (15.22)$$

where $\rho_{0,1}$ is chosen in such a way that:

$$\left(\frac{\rho_{0,1}}{\rho_0} \right)^J \left(1 + J^{\alpha-1} \varepsilon c_{0,0} e^{-(\delta_0 + \Gamma_0(\lambda))} (1 + c_2 \varepsilon)^{J-2} \right) \leq 1 . \quad (15.23)$$

15.3 Proof of Lemma 4.2:

Assume for simplicity $\Psi(R)$ denotes a S -tangent vector, like for instance the null component β , with O a rotation generator vector,

$$\mathcal{L}_O \Psi(\cdot) = \nabla_O \Psi(\cdot) - \Psi_b \nabla \cdot O^b \equiv \nabla_O \Psi(\cdot) - \Psi_b c^b(\cdot) \quad (15.24)$$

where

$$c^b(\cdot) = \nabla \cdot O^b .$$

Recalling theorem 9.6, the following estimates for the initial data hold

$$|r^{(J+1-\frac{2}{p})} \nabla^{J-1} {}^{(i)} O|_{\infty, S} \leq c \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_0^J} \right) , \quad (15.25)$$

$$\mathcal{L}_O \Psi(\cdot) = \nabla_O \Psi(\cdot) - (\Psi \cdot c)(\cdot) , \quad (15.26)$$

we write symbolically as

$$\mathcal{L}_O \Psi = (\nabla_O - c) \Psi \quad (15.27)$$

and

$$\mathcal{L}_O^{J-1} \Psi = (\nabla_O - c)^{J-1} \Psi = \sum_{k=0}^{J-1} \binom{J-1}{k} \nabla_O^k c^{J-1-k} \Psi \quad (15.28)$$

where in the right hand side we have to be careful at the position of the ∇_O 's as they can operate on the c 's or on the Ψ . Therefore fixed k one has to interpret the right hand side as made by $J - k$ slots, each one on the left of one c and the last slot on the left of the Ψ . In these $J - k$ slots one has to distribute k ∇_O 's, each one operating exclusively on the tensor c at its immediate right, or on Ψ for the last slot. Computing all the possible ways of distributing these ∇_O 's one is considering more terms of the real ones, but this over estimate should not be harmful. Therefore, omitting the minus signs as at the end we are doing an upper bound norm estimate, we write the following expression¹¹⁸

$$\Psi(\mathcal{L}_O^{J-1}(\cdot)) \text{ " = " } \sum_{k=0}^{J-1} \binom{J-1}{k} \sum_{\substack{\gamma_s=k \\ \gamma_1, \gamma_2, \dots, \gamma_{J-k}}} \frac{k!}{\gamma_1! \gamma_2! \dots \gamma_{J-k}!} (\nabla_O^{\gamma_1} \hat{c})(\nabla_O^{\gamma_2} \hat{c}) \dots (\nabla_O^{\gamma_{J-1-k}} \hat{c})(\nabla_O^{\gamma_{J-k}} \Psi)$$

and the norm estimate is

$$\leq \frac{J!}{J^\alpha} \left[\sum_{k=0}^{J-1} \frac{J^{\alpha-1}}{(J-1-k)!} \sum_{\substack{\gamma_s=k \\ \gamma_1, \gamma_2, \dots, \gamma_{J-k}}} \frac{1}{\gamma_1! \gamma_2! \dots \gamma_{J-k}!} \left| (\nabla_O^{\gamma_1} \hat{c})(\nabla_O^{\gamma_2} \hat{c}) \dots (\nabla_O^{\gamma_{J-1-k}} \hat{c})(\nabla_O^{\gamma_{J-k}} \Psi) \right|_{p,S} \right]$$

In order to estimate $\nabla_O c$ we apply lemma 4.1 to c and recalling theorem 9.6, it follows that with a suitable c

$$|\nabla_O^\gamma c|_{\infty, S} \leq \left(\frac{(\gamma+1)!}{(\gamma+1)^\alpha} \frac{e^{((\gamma+1)-2)(\delta_0 + \Gamma_0(\lambda))}}{\rho_{0,1}^{\gamma+1}} \right) \quad (15.29)$$

Now notice that at least one among the γ_i is greater than $\frac{k}{J-k}$ hence, without lost of generality, we can suppose $\gamma_{J-k} \geq \frac{k}{J-k}$

$$\begin{aligned} & \frac{J!}{J^\alpha} \left[\sum_{k=0}^{J-1} \frac{J^{\alpha-1}}{(J-1-k)!} \sum_{\substack{\gamma_s=k \\ \gamma_1, \gamma_2, \dots, \gamma_{J-k}}} \frac{1}{\gamma_1! \gamma_2! \dots \gamma_{J-k}!} \left| (\nabla_O^{\gamma_1} \hat{c})(\nabla_O^{\gamma_2} \hat{c}) \dots (\nabla_O^{\gamma_{J-1-k}} \hat{c})(\nabla_O^{\gamma_{J-k}} \Psi) \right|_{p,S} \right] \\ & \leq \frac{J!}{J^\alpha} \left[\sum_{k=0}^{J-1} \frac{J^{\alpha-1}}{(J-1-k)!} \sum_{\substack{\gamma_s=k \\ \gamma_1, \gamma_2, \dots, \gamma_{J-k}}} \frac{1}{\gamma_1! \gamma_2! \dots \gamma_{J-k}!} \left| (\nabla_O^{\gamma_1} \hat{c})(\nabla_O^{\gamma_2} \hat{c}) \dots (\nabla_O^{\gamma_{J-1-k}} \hat{c}) \right|_{\infty, S} \left| (\nabla_O^{\gamma_{J-k}} \Psi) \right|_{p,S} \right] \\ & \leq \frac{J!}{J^\alpha} \left[\sum_{k=0}^{J-1} \frac{J^{\alpha-1}}{(J-1-k)!} \sum_{\substack{\gamma_s=k \\ \gamma_1, \gamma_2, \dots, \gamma_{J-k}}} c^{J-k} \frac{(\gamma_1+1)}{(\gamma_1+1)^\alpha} \frac{(\gamma_2+1)}{(\gamma_2+1)^\alpha} \dots \frac{(\gamma_{J-1-k}+1)}{(\gamma_{J-1-k}+1)^\alpha} \frac{(\gamma_{J-k}+1)}{(\gamma_{J-k}+1)^\alpha} \frac{e^{(k-(J-k))(\delta+\Gamma)}}{\rho_{0,1}^J} \right] \\ & \leq \frac{J!}{J^\alpha} \left(\frac{e^{(J-2)(\delta+\Gamma)}}{\rho_{0,1}^J} \right) \left[\sum_{k=0}^{J-1} c^{J-k} \frac{J^{\alpha-1}}{(J-1-k)!} \right] \end{aligned}$$

¹¹⁸To avoid too cumbersome notations we avoided the weight factors in the norm estimates; following the previous considerations it will be easy to reinsert them.

$$\begin{aligned}
& \sum_{\gamma_1, \gamma_2, \dots, \gamma_{J-k}} \frac{1}{(\gamma_1 + 1)^{\alpha-1}} \frac{1}{(\gamma_2 + 1)^{\alpha-1}} \cdots \frac{1}{(\gamma_{J-1-k} + 1)^{\alpha-1}} \frac{1}{(\gamma_{J-k} + 1)^{\alpha-1}} e^{-2J(\delta+\Gamma)} \Bigg] \\
& \leq \frac{J!}{J^\alpha} \left(\frac{e^{(J-2)(\delta+\Gamma)}}{\rho_{0,1}^J} \right) \left[\sum_{k=0}^{J-1} \frac{J^{\alpha-1} (J-k)^{\alpha-1}}{k^{\alpha-1}} \frac{c^{J-k} e^{-2(J-1)(\delta+\Gamma)}}{(J-1-k)! \rho_{0,1}^{J-k-1}} \right. \\
& \quad \left. \left(\sum_{\gamma_1, \gamma_2, \dots, \gamma_{J-k}} \frac{1}{(\gamma_1 + 1)^{\alpha-1}} \frac{1}{(\gamma_2 + 1)^{\alpha-1}} \cdots \frac{1}{(\gamma_{J-1-k} + 1)^{\alpha-1}} \right) \right] \\
& \leq c \frac{J!}{J^\alpha} \left(\frac{e^{(J-2)(\delta+\Gamma)}}{\rho_{0,1}^J} \right) \left[\sum_{k=0}^{J-1} \frac{J^{\alpha-1} (J-k)^{\alpha-1}}{k^{\alpha-1}} \frac{2^{J-k} c^{J-k} e^{-2(J-1-k)(\delta+\Gamma)}}{(J-1)!} \right]
\end{aligned} \tag{15.30}$$

provided $\alpha > 3^{119}$. We rearrange the terms in the following way

$$\begin{aligned}
& \leq c \frac{J!}{J^\alpha} \left(\frac{e^{(J-2)(\delta+\Gamma)}}{\rho_{0,1}^J} \right) \left[\sum_{k=0}^{J-1} \frac{J^{\alpha-1}}{k^{\alpha-1} (J-1-k)!} \frac{(J-k)^{\alpha-1} c^{J-k} e^{-2(J-1-k)(\delta+\Gamma)}}{\rho_{0,1}^{J-k-1}} \right] \\
& \leq c' \frac{J!}{J^\alpha} \left(\frac{e^{(J-2)(\delta+\Gamma)}}{\rho_{0,1}^J} \right) \left[\sum_{k=0}^{J-1} \frac{J^{\alpha-1}}{k^{\alpha-1} (J-1-k)!} \right] \\
& \leq C \frac{J!}{J^\alpha} \left(\frac{e^{(J-2)(\delta+\Gamma)}}{\rho_{0,1}^J} \right),
\end{aligned} \tag{15.31}$$

provided δ sufficiently large. Clearly the same lemma can be used to pass from $\hat{\mathcal{L}}_O(\Psi(R))$ to $\mathcal{L}_O(\Psi(R))$ and from $\mathcal{L}_O(\Psi(R))$ to $\Psi(\mathcal{L}_O R)$

Let us write down symbolically the formula to pass from $\mathcal{L}_O \Psi(R)$ to $\Psi(\mathcal{L}_O R)$, see [?] proposition 5.1.1 equation 5.1.4 as example,

$$\psi(\mathcal{L}_O R) = (\mathcal{L}_O - \hat{c})\Psi(R), \tag{15.32}$$

where \hat{c} is a combination of connection coefficients.

At its turn we can write

$$\Psi(\hat{\mathcal{L}}_O R) = (\mathcal{L}_O - \tilde{c})\Psi(R), \tag{15.33}$$

where \tilde{c} is a combination of connection coefficients, see [Kl-Ni2], proposition 5.1.1. equation 5.1.6 as example.

¹¹⁹We need $\alpha > 3$ due to the possible estimate of $\left| (\mathcal{L}_O^{\gamma_{J-k}} \Psi) \right|_\infty$ which give a term $\frac{1}{(\gamma_{J-k})^{\alpha-2}}$

15.4 Theorem 10.1, the Proof of the boundedness of the \mathcal{Q} norms. The estimate of the Error

The Proof of the boundedness is more complicated than the one discussed in [Kl-Ni2], due to the presence of terms associated to all values of J .

The way to prove our result mimics what done in [Kl-Ni2] and more in general each time we have a global existence proof, namely we assume the existence of the solution satisfying some specific properties in a region we consider the largest possible and, subsequently, we prove that this region can be extended. This implies, to avoid a contradiction, that the assumed “largest region” is in fact the whole space where the problem is well defined. This argument requires, not to be empty, that one could prove, usually via a locally existence result, that a solution with the “specific properties” does exist, at least in a small region. Therefore we have to look carefully at the following two points:

- i) Which are the specific properties we have to assume.
- ii) How we extend the assumed “largest possible region”.

The second point is the heart of the problem, but the choice of the first is crucial to allow the second one.

The specific properties:

We assume that in the “largest possible region” \mathcal{K} :

- a) The norms of the connection coefficients with all their derivatives satisfy some bounds which are those expressed in the initial sections and we are going to recall later on, with a well definite constant in front which we assume small, ε , for all derivatives up to the J_0 order.
- b) The norms of the Riemann components with all their derivatives satisfy some bounds which are those expressed in the initial sections and we are going to recall later on, with a well definite constant in front which we assume small, $O(\varepsilon)$, for all derivatives up to the J_0 order.
- c) The “largest possible region” has a diamond shape and the last slice is $C_* \equiv C(\nu_*; [\lambda_0, \lambda_1])$. The energy-type norms \mathcal{Q} defined on the “largest possible region” have to satisfy the following bounds, for any J and any $\lambda \in [\lambda_0, \bar{\lambda}]$, and $\nu \in [\nu_0, \bar{\nu}]$, with c sufficiently small

$$\left[\sum_{H=2}^J (\mathcal{Q}^{(H-2)}(\lambda, [\nu_0, \nu]) + \underline{\mathcal{Q}}^{(H-2)}([\lambda_0, \lambda], \nu) \right] \leq c \left[\sum_{H=2}^J (\mathcal{Q}_{(0),2}^{(H-2)}(\lambda_0, [\nu_0, \nu]) + \underline{\mathcal{Q}}_{(0),2}^{(H-2)}([\lambda_0, \lambda], \nu_0) \right]. \quad (15.34)$$

Remark:

We expect that requirement b) is redundant as all these norm estimates, provided requirement a) is satisfied, can be derived from the boundedness of the energy-type norm estimates, requirement c).

How to extend the region:

As said in subsection 4.1 we have to prove the following inequality in \mathcal{K} (here we write only the one in λ , the one in ν is similar):

$$Q(C(\lambda)) = Q(C_0) + Error = Q(C_0) + \int_0^\lambda d\lambda' |F(\mathcal{O})| Q(C(\lambda'))$$

If we can prove the *Error* is sufficiently small to lower the constant $C^{(1)}$, of the inductive estimates in \mathcal{K} for the null Riemann components, estimate 4.22, we can extend the region \mathcal{K} and obtain the global existence. This result can be obtained estimating carefully the *Error* of the $Q^{(J-2)}$ norms for every J . The *Error* term, we will divide it in \mathcal{E}_1 and \mathcal{E}_2 , we have to estimate have the following structure, see [Kl-Ni2],

$$\begin{aligned} \mathcal{E}_1(u, \underline{u}) &= \int_{V(u, \underline{u})} Div Q(\hat{\mathcal{L}}_T \tilde{W})_{\beta\gamma\delta} (\bar{K}^\beta \bar{K}^\gamma \bar{K}^\delta) \\ &\quad + \int_{V(u, \underline{u})} Div Q(\hat{\mathcal{L}}_O \tilde{W})_{\beta\gamma\delta} (\bar{K}^\beta \bar{K}^\gamma T^\delta) \\ &\quad + \frac{3}{2} \int_{V(u, \underline{u})} Q(\hat{\mathcal{L}}_T \tilde{W})_{\alpha\beta\gamma\delta} ({}^{(\bar{K})} \pi^{\alpha\beta} \bar{K}^\gamma \bar{K}^\delta) \\ &\quad + \int_{V(u, \underline{u})} Q(\hat{\mathcal{L}}_O \tilde{W})_{\alpha\beta\gamma\delta} ({}^{(\bar{K})} \pi^{\alpha\beta} \bar{K}^\gamma T^\delta) \\ &\quad + \frac{1}{2} \int_{V(u, \underline{u})} Q(\hat{\mathcal{L}}_O \tilde{W})_{\alpha\beta\gamma\delta} ({}^{(T)} \pi^{\alpha\beta} \bar{K}^\gamma \bar{K}^\delta) \end{aligned} \quad (15.35)$$

$$\begin{aligned} \mathcal{E}_2(u, \underline{u}) &= \int_{V(u, \underline{u})} Div Q(\hat{\mathcal{L}}_O^2 \tilde{W})_{\beta\gamma\delta} (\bar{K}^\beta \bar{K}^\gamma T^\delta) \\ &\quad + \int_{V(u, \underline{u})} Div Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T \tilde{W})_{\beta\gamma\delta} (\bar{K}^\beta \bar{K}^\gamma \bar{K}^\delta) \\ &\quad + \int_{V(u, \underline{u})} Div Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T \tilde{W})_{\beta\gamma\delta} (\bar{K}^\beta \bar{K}^\gamma \bar{K}^\delta) \\ &\quad + \int_{V(u, \underline{u})} Q(\hat{\mathcal{L}}_O^2 \tilde{W})_{\alpha\beta\gamma\delta} ({}^{(\bar{K})} \pi^{\alpha\beta} \bar{K}^\gamma T^\delta) \\ &\quad + \frac{1}{2} \int_{V(u, \underline{u})} Q(\hat{\mathcal{L}}_O^2 \tilde{W})_{\alpha\beta\gamma\delta} ({}^{(T)} \pi^{\alpha\beta} \bar{K}^\gamma \bar{K}^\delta) \\ &\quad + \frac{3}{2} \int_{V(u, \underline{u})} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T \tilde{W})_{\alpha\beta\gamma\delta} ({}^{(\bar{K})} \pi^{\alpha\beta} \bar{K}^\gamma \bar{K}^\delta) \\ &\quad + \frac{3}{2} \int_{V(u, \underline{u})} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T \tilde{W})_{\alpha\beta\gamma\delta} ({}^{(\bar{K})} \pi^{\alpha\beta} \bar{K}^\gamma \bar{K}^\delta), \end{aligned} \quad (15.36)$$

where $\tilde{W} = \hat{\mathcal{L}}_O^{J-2} W$ and W (but not \tilde{W}) satisfies the Bianchi identities.

Looking at these expressions it follows that the terms without the divergence can be treated exactly as in [Kl-Ni2] as the presence of \tilde{W} instead of W does not play any role, in fact the only property which will be used is that \tilde{W} is a Weyl tensor exactly as W . The situation is, viceversa, different for the derived terms as in the present case $D^\mu \tilde{W}_{\mu\nu\rho\sigma}$ is different from zero while $D^\mu W_{\mu\nu\rho\sigma} = 0$. This implies that some more terms are present in the error.

Let us look at it in greater detail considering separately the parts relative to $\mathcal{E}_1(u, \underline{u})$,

$$\begin{aligned} & \int_{V(u, \underline{u})} \text{Div} Q(\hat{\mathcal{L}}_O \tilde{W})_{\beta\gamma\delta} (\bar{K}^\beta \bar{K}^\gamma T^\delta) \\ & \int_{V(u, \underline{u})} \text{Div} Q(\hat{\mathcal{L}}_T \tilde{W})_{\beta\gamma\delta} (\bar{K}^\beta \bar{K}^\gamma \bar{K}^\delta) \end{aligned} \quad (15.37)$$

and those relative to $\mathcal{E}_2(u, \underline{u})$,

$$\begin{aligned} & \int_{V(u, \underline{u})} \text{Div} Q(\hat{\mathcal{L}}_O^2 \tilde{W})_{\beta\gamma\delta} (\bar{K}^\beta \bar{K}^\gamma T^\delta) \\ & \int_{V(u, \underline{u})} \text{Div} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T \tilde{W})_{\beta\gamma\delta} (\bar{K}^\beta \bar{K}^\gamma \bar{K}^\delta) \\ & \int_{V(u, \underline{u})} \text{Div} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T \tilde{W})_{\beta\gamma\delta} (\bar{K}^\beta \bar{K}^\gamma \bar{K}^\delta) . \end{aligned} \quad (15.38)$$

For simplicity we restrict to the first terms of $\mathcal{E}_1(u, \underline{u})$ and of $\mathcal{E}_2(u, \underline{u})$ as we expect that the remaining terms can be treated in the same way,

$$\begin{aligned} & \int_{V(u, \underline{u})} \text{Div} Q(\hat{\mathcal{L}}_O \tilde{W}^{(J-2)})_{\beta\gamma\delta} (\bar{K}^\beta \bar{K}^\gamma T^\delta) \\ & \int_{V(u, \underline{u})} \text{Div} Q(\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\beta\gamma\delta} (\bar{K}^\beta \bar{K}^\gamma T^\delta) . \end{aligned}$$

From Proposition 7.1.1 of [Ch-Kl] we have the following expressions

$$\begin{aligned} & \text{Div} Q(\hat{\mathcal{L}}_O \tilde{W}^{(J-2)})_{\beta\gamma\delta} = \\ & = (\hat{\mathcal{L}}_O \tilde{W}^{(J-2)})_{\beta\delta}^{\mu\nu} \mathbf{D}^\alpha (\hat{\mathcal{L}}_O \tilde{W}^{(J-2)})_{\alpha\mu\gamma\nu} + (\hat{\mathcal{L}}_O \tilde{W}^{(J-2)})_{\beta\gamma}^{\mu\nu} \mathbf{D}^\alpha (\hat{\mathcal{L}}_O \tilde{W}^{(J-2)})_{\alpha\mu\delta\nu} \\ & + * (\hat{\mathcal{L}}_O \tilde{W}^{(J-2)})_{\beta\delta}^{\mu\nu} \mathbf{D}^\alpha (\hat{\mathcal{L}}_O * \tilde{W}^{(J-2)})_{\alpha\mu\gamma\nu} + * (\hat{\mathcal{L}}_O \tilde{W}^{(J-2)})_{\beta\gamma}^{\mu\nu} \mathbf{D}^\alpha (\hat{\mathcal{L}}_O * \tilde{W}^{(J-2)})_{\alpha\mu\delta\nu} \end{aligned} \quad (15.39)$$

$$\begin{aligned} & \text{Div} Q(\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\beta\gamma\delta} = \\ & = (\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\beta\delta}^{\mu\nu} \mathbf{D}^\alpha (\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\alpha\mu\gamma\nu} + (\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\beta\gamma}^{\mu\nu} \mathbf{D}^\alpha (\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\alpha\mu\delta\nu} \\ & + * (\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\beta\delta}^{\mu\nu} \mathbf{D}^\alpha (\hat{\mathcal{L}}_O^2 * \tilde{W}^{(J-2)})_{\alpha\mu\gamma\nu} + * (\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\beta\gamma}^{\mu\nu} \mathbf{D}^\alpha (\hat{\mathcal{L}}_O^2 * \tilde{W}^{(J-2)})_{\alpha\mu\delta\nu} \end{aligned} \quad (15.40)$$

We consider the term associated to the \mathcal{E}_2 error as,¹²⁰ once we prove the right estimate for this term, the estimate for the term associated to \mathcal{E}_1 follows. Therefore we consider the first term of 15.40,

$$Div Q(\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\beta\gamma\delta} = (\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\beta}^{\mu}{}_{\delta}^{\nu} \mathbf{D}^{\alpha} (\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\alpha\mu\gamma\nu} \quad (15.41)$$

and the part of the error term \mathcal{E}_2 we consider is,

$$\int_{V_{(\lambda,\nu)}} (\mathbf{D}^{\mu} \hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\mu\rho\gamma\sigma} (\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\beta}^{\rho}{}_{\delta}^{\sigma} (\bar{K}^{\beta} \bar{K}^{\gamma} T^{\delta}) . \quad (15.42)$$

Considering only this part as “error term” we can write,

$$\begin{aligned} \mathcal{Q}_2^{(J-2)}(\lambda, \nu) &= \mathcal{Q}_{(0),2}^{(J-2)} + \int_{V_{(\lambda,\nu)}} (\mathbf{D}^{\mu} \hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\mu\rho\gamma\sigma} (\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\beta}^{\rho}{}_{\delta}^{\sigma} (\bar{K}^{\beta} \bar{K}^{\gamma} T^{\delta}) \\ &\quad + \int_{V_{(\lambda,\nu)}} (\mathbf{D}^{\mu} \hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\mu\rho\delta\sigma} (\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\beta}^{\rho}{}_{\gamma}^{\sigma} (\bar{K}^{\beta} \bar{K}^{\gamma} T^{\delta}) \end{aligned} \quad (15.43)$$

and the problem is reduced to showing that the second term and the third one can be bounded by $\varepsilon_1(\mathcal{Q}_1^{(J-2)} + \mathcal{Q}_2^{(J-2)})(\lambda, \nu)$ where with ε_1 we mean a small quantity (not a priori related to ε). The two parts of the error we are considering are basically equal and, therefore, we estimate only the first one.

We start computing $(\mathbf{D}^{\mu} \hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\mu\beta\gamma\delta}$: From Proposition 7.1.2 in [Ch-Kl] it follows,

$$\begin{aligned} (\mathbf{D}^{\mu} \hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\mu\beta\gamma\delta} &= \mathbf{D}^{\mu} (\hat{\mathcal{L}}_O \tilde{W}^{(J-1)})_{\mu\beta\gamma\delta} \\ &= \mathcal{L}_O \mathbf{D}^{\mu} \tilde{W}_{\mu\beta\gamma\delta}^{(J-1)} + \left[-\frac{1}{2} \left({}^{(O)}\hat{\pi}^{\rho}{}_{\beta} \mathbf{D}^{\mu} (\tilde{W}^{(J-1)})_{\mu\rho\gamma\delta} + {}^{(O)}\hat{\pi}^{\rho}{}_{\gamma} \mathbf{D}^{\mu} (\tilde{W}^{(J-1)})_{\mu\beta\rho\delta} + {}^{(O)}\hat{\pi}^{\rho}{}_{\delta} \mathbf{D}^{\mu} (\tilde{W}^{(J-1)})_{\mu\beta\gamma\rho} \right) \right. \\ &\quad \left. + \frac{1}{8} (\text{tr}^{(O)} \pi) \mathbf{D}^{\mu} (\tilde{W}^{(J-1)})_{\mu\beta\gamma\delta} \right] \\ &\quad + \frac{1}{2} {}^{(O)}\hat{\pi}^{\mu\nu} \mathbf{D}_{\nu} \tilde{W}_{\mu\beta\gamma\delta}^{(J-1)} + \frac{1}{2} \mathbf{D}^{\alpha} ({}^{(O)}\hat{\pi}_{\alpha\lambda} \tilde{W}_{\beta\gamma\delta}^{(J-1)\lambda}) + \frac{1}{2} \left\{ (\mathbf{D}_{\beta} {}^{(O)}\hat{\pi}_{\mu\lambda} - \mathbf{D}_{\lambda} {}^{(O)}\hat{\pi}_{\mu\beta}) \tilde{W}_{\gamma\delta}^{(J-1)\mu\lambda} + \dots \right\} \\ &= \mathcal{L}_O \mathbf{D}^{\mu} \tilde{W}_{\mu\beta\gamma\delta}^{(J-1)} + \left[-\frac{1}{2} \left({}^{(O)}\hat{\pi}^{\rho}{}_{\beta} \mathbf{D}^{\mu} (\tilde{W}^{(J-1)})_{\mu\rho\gamma\delta} + {}^{(O)}\hat{\pi}^{\rho}{}_{\gamma} \mathbf{D}^{\mu} (\tilde{W}^{(J-1)})_{\mu\beta\rho\delta} + {}^{(O)}\hat{\pi}^{\rho}{}_{\delta} \mathbf{D}^{\mu} (\tilde{W}^{(J-1)})_{\mu\beta\gamma\rho} \right) \right. \\ &\quad \left. + \frac{1}{8} (\text{tr}^{(O)} \pi) \mathbf{D}^{\mu} (\tilde{W}^{(J-1)})_{\mu\beta\gamma\delta} + {}^{(O)}\hat{\pi}^{\mu\nu} \mathbf{D}_{\nu} \tilde{W}_{\mu\beta\gamma\delta}^{(J-1)} \right] \\ &\quad + \left\{ {}^{(O)}p_{\lambda} \tilde{W}_{\beta\gamma\delta}^{(J-1)\lambda} + \left({}^{(O)}q_{\alpha\beta\lambda} \tilde{W}_{\gamma\delta}^{(J-1)\alpha\lambda} + {}^{(O)}q_{\alpha\gamma\lambda} \tilde{W}_{\beta\delta}^{(J-1)\alpha\lambda} + {}^{(O)}q_{\alpha\delta\lambda} \tilde{W}_{\beta\gamma}^{(J-1)\alpha\lambda} \right) \right\} \end{aligned} \quad (15.44)$$

where ${}^{(O)}p_{\lambda}$ and the ${}^{(O)}q_{\alpha\delta\lambda}$ are defined in [Kl-Ni2].

Observe that the first term has $(J-1) + 2$ derivations, the terms in square brackets have $(J-1) + 1$ derivations and the terms in the curly brackets have

¹²⁰In fact it turns out that the estimate for \mathcal{E}_1 can also be done in a easier way.

$(J-1)$ derivations. It is the first term we cannot estimate with $\mathcal{Q}_2^{(J-2)}$ while the second and third term can be easily estimated in terms of $\mathcal{Q}_2^{(J-2)}$. We have, therefore, to iterate the procedure computing again as before

$$\mathbf{D}^\mu \tilde{W}_{\mu\beta\gamma\delta}^{(J-1)} = (\mathbf{D}^\mu \hat{\mathcal{L}}_O \tilde{W}^{(J-2)})_{\mu\beta\gamma\delta}$$

and then repeating the procedure up to the moment we have

$$\begin{aligned} \mathcal{L}_O^{J-1} \mathbf{D}^\mu \tilde{W}_{\mu\beta\gamma\delta}^{(1)} &= \mathcal{L}_O^{J-1} (\mathbf{D}^\mu \hat{\mathcal{L}}_O \tilde{W}^{(0)})_{\mu\beta\gamma\delta} = \mathcal{L}_O^{J-1} (\mathbf{D}^\mu \hat{\mathcal{L}}_O W)_{\mu\beta\gamma\delta} \\ &= \mathcal{L}_O^{J-1} ([\mathbf{D}^\mu, \hat{\mathcal{L}}_O] W)_{\mu\beta\gamma\delta} + \mathcal{L}_O^{J-1} (\hat{\mathcal{L}}_O \mathbf{D}^\mu W)_{\mu\beta\gamma\delta} = \mathcal{L}_O^{J-1} ([\mathbf{D}^\mu, \hat{\mathcal{L}}_O] W)_{\mu\beta\gamma\delta}, \end{aligned} \quad (15.45)$$

as $W = \tilde{W}^{(0)}$ has divergence zero. Therefore we have to repeat the procedure J times which implies that at every step the terms in square brackets and curly brackets reproduce themselves, obtaining

$$\begin{aligned} (\mathbf{D}^\mu \hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\mu\beta\gamma\delta} &= (\mathbf{D}^\mu \hat{\mathcal{L}}_O \tilde{W}^{(J-1)})_{\mu\beta\gamma\delta} = \\ &= \sum_{k=0}^{J-1} \mathcal{L}_O^k \left[-\frac{1}{2} \left({}^{(O)}\hat{\pi}^\rho_\beta \mathbf{D}^\mu (\tilde{W}^{(J-1-k)})_{\mu\rho\gamma\delta} + {}^{(O)}\hat{\pi}^\rho_\gamma \mathbf{D}^\mu (\tilde{W}^{(J-1-k)})_{\mu\beta\rho\delta} + {}^{(O)}\hat{\pi}^\rho_\delta \mathbf{D}^\mu (\tilde{W}^{(J-1-k)})_{\mu\beta\gamma\rho} \right) \right. \\ &\quad \left. + \frac{1}{8} (\text{tr}^{(O)} \pi) \mathbf{D}^\mu (\tilde{W}^{(J-1-k)})_{\mu\beta\gamma\delta} + {}^{(O)}\hat{\pi}^{\mu\nu} \mathbf{D}_\nu \tilde{W}_{\mu\beta\gamma\delta}^{(J-1-k)} \right] \\ &+ \sum_{k=0}^{J-1} \mathcal{L}_O^k \left\{ {}^{(O)}p_\lambda \tilde{W}^{(J-1-k)\lambda}_{\beta\gamma\delta} + \left({}^{(O)}q_{\alpha\beta\lambda} \tilde{W}^{(J-1-k)\alpha\lambda}_{\gamma\delta} + {}^{(O)}q_{\alpha\gamma\lambda} \tilde{W}^{(J-1-k)\alpha\lambda}_{\beta\delta} + {}^{(O)}q_{\alpha\delta\lambda} \tilde{W}^{(J-1-k)\alpha\lambda}_{\beta\gamma} \right) \right\} \end{aligned} \quad (15.46)$$

which we write symbolically, omitting the constants, as

$$\begin{aligned} \sum_{k=0}^{J-2} \mathcal{L}_O^k \left({}^{(O)}\pi^\rho_\beta \mathbf{D}^\mu (\tilde{W}^{(J-1-k)})_{\mu\rho\gamma\delta} \right) &+ \sum_{k=0}^{J-1} \mathcal{L}_O^k \left({}^{(O)}\hat{\pi}^{\mu\nu} \mathbf{D}_\nu \tilde{W}_{\mu\beta\gamma\delta}^{(J-1-k)} \right) \\ &+ \sum_{k=0}^{J-1} \mathcal{L}_O^k \left({}^{(O)}\tilde{q}_{\alpha\beta\lambda} \tilde{W}^{(J-1-k)\alpha\lambda}_{\gamma\delta} \right), \end{aligned} \quad (15.47)$$

where

$$\tilde{q}_{\alpha\beta\lambda} = \delta_{\alpha\beta} {}^{(O)}p_\lambda \quad \text{or} \quad {}^{(O)}q_{\alpha\beta\lambda}.$$

Let us estimate the first sum, the second one should be estimated in the same way, the third one will be considered later on. Before doing this estimate let us look at what we would like to get. The error, considering only the first sum, has the form

$$\begin{aligned} &\int_{V_{(\lambda,\nu)}} (\mathbf{D}^\mu \hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\mu\rho\gamma\sigma} (\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\beta\delta}^{\rho\sigma} (\bar{K}^\beta \bar{K}^\gamma T^\delta) \\ &= \sum_{k=0}^{J-2} \int_{V_{(\lambda,\nu)}} \left(\mathcal{L}_O^k {}^{(O)}\pi^\tau_\rho \mathbf{D}^\mu (\tilde{W}^{(J-1-k)})_{\mu\tau\gamma\sigma} \right) (\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\beta\delta}^{\rho\sigma} (\bar{K}^\beta \bar{K}^\gamma T^\delta). \end{aligned} \quad (15.48)$$

Therefore neglecting all the other parts of the error which we will consider later on we have to deal with the following inequality,

$$\begin{aligned} & \mathcal{Q}_2^{(J-2)}(\lambda, \nu) + \underline{\mathcal{Q}}_2^{(J-2)}(\lambda, \nu) = \mathcal{Q}_{(0),2}^{(J-2)}(\lambda, \nu) \\ & + \sum_{k=0}^{J-2} \int_{V_{(\lambda, \nu)}} \left(\mathcal{L}_O^k {}^{(O)}\pi^\tau_\rho \mathbf{D}^\mu (\tilde{W}^{(J-1-k)})_{\mu\tau\gamma\sigma} \right) (\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\beta}^{\rho}{}_{\delta}^{\sigma} (\bar{K}^\beta \bar{K}^\gamma T^\delta) \end{aligned}$$

where

$$\mathcal{Q}_{(0),2}^{(J-2)}(\lambda, \nu) = \mathcal{Q}_{(0),2}^{(J-2)}(\lambda_0, \nu) + \underline{\mathcal{Q}}_{(0),2}^{(J-2)}(\lambda, \nu_0). \quad (15.49)$$

To estimate this part of the error we proceed as done in [Kl-Ni2], we consider one term of the sum of terms in the integrand

$$\sum_{k=0}^{J-2} \int_{V_{(\lambda, \nu)}} \tau_+^4 \left(\mathcal{L}_O^k {}^{(O)}\pi^\tau_\rho \mathbf{D}^\mu (\tilde{W}^{(J-1-k)})_{\mu\tau 4\sigma} \right) (\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{4}^{\rho}{}_{4}^{\sigma} \quad (15.50)$$

which at its turn can be decomposed in two terms

$$\begin{aligned} & \sum_{k=0}^{J-2} \int_{V_{(\lambda, \nu)}} \tau_+^4 \left(\mathcal{L}_O^k {}^{(O)}\pi^\tau_\rho \mathbf{D}^\mu (\tilde{W}^{(J-1-k)})_{\mu\tau 4\sigma} \right) (\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{4}^{\rho}{}_{4}^{\sigma} \\ & = \sum_{k=0}^{J-2} \int_{V_{(\lambda, \nu)}} \tau_+^4 \left(\mathcal{L}_O^k {}^{(O)}\pi^\tau_\rho \mathbf{D}(\tilde{W}^{(J-1-k)}) \right) \cdot \alpha(\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)}) \\ & + \sum_{k=0}^{J-2} \int_{V_{(\lambda, \nu)}} \tau_+^4 \left(\mathcal{L}_O^k {}^{(O)}\pi^\tau_\rho \mathbf{D}(\tilde{W}^{(J-1-k)}) \right) \cdot \beta(\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)}) \end{aligned} \quad (15.51)$$

The two terms have the same structure therefore we estimate only the first one. Let us consider a generic term of the first sum

$$\begin{aligned} & \int_{V_{(\lambda, \nu)}} \tau_+^4 \left(\mathcal{L}_O^k {}^{(O)}\pi^\tau_\rho \mathbf{D}(\tilde{W}^{(J-1-k)}) \right) \cdot \alpha(\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)}) \\ & = \sum_{l=0}^k \binom{k}{l} \int_{V_{(\lambda, \nu)}} \tau_+^4 \left((\mathcal{L}_O^l {}^{(O)}\pi^\tau_\rho) \hat{\mathcal{L}}_O^{k-l} \mathbf{D}(\tilde{W}^{(J-1-k)}) \right) \cdot \alpha(\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)}) \\ & = \sum_{l=0}^k \binom{k}{l} \int_{\lambda_0}^\lambda d\lambda' \int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 \left((\mathcal{L}_O^l {}^{(O)}\pi^\tau_\rho) \hat{\mathcal{L}}_O^{k-l} \mathbf{D}(\tilde{W}^{(J-1-k)}) \right) \cdot \alpha(\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)}) \\ & = \sum_{l=0}^k \binom{k}{l} \int_{\lambda_0}^\lambda d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^l {}^{(O)}\pi^\tau_\rho) \hat{\mathcal{L}}_O^{k-l} \mathbf{D}(\tilde{W}^{(J-1-k)})|^2 \right)^{\frac{1}{2}} \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |\alpha(\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})|^2 \right)^{\frac{1}{2}} \\ & \leq \left(\sup_{\lambda' \in [\lambda_0, \lambda]} \mathcal{Q}_2^{(J-2)}(\lambda', \nu) \right)^{\frac{1}{2}} \left[\sum_{l=0}^k \binom{k}{l} \int_{\lambda_0}^\lambda d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^l {}^{(O)}\pi^\tau_\rho) \hat{\mathcal{L}}_O^{k-l} \mathbf{D}(\tilde{W}^{(J-1-k)})|^2 \right)^{\frac{1}{2}} \right]. \end{aligned} \quad (15.52)$$

Therefore assuming for the moment this as the only error contribution we have

$$\mathcal{Q}_2^{(J-2)}(\lambda, \nu) + \underline{\mathcal{Q}}_2^{(J-2)}(\lambda, \nu) \leq \mathcal{Q}_{(0),2}^{(J-2)}(\lambda, \nu)$$

$$+ \left(\sup_{\lambda' \in [\lambda_0, \lambda]} \mathcal{Q}_2^{(J-2)}(\lambda', \nu) \right)^{\frac{1}{2}} \sum_{k=0}^{J-2} \left[\sum_{l=0}^k \binom{k}{l} \int_{\lambda_0}^{\lambda} d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^l)^{(O)} \pi_{\rho}^{\tau} \hat{\mathcal{L}}_O^{k-l} \mathbf{D}(\tilde{W}^{(J-1-k)})|^2 \right)^{\frac{1}{2}} \right].$$

Let us state now the estimate assumptions for $\mathcal{Q}_2^{(H)}(\lambda, \nu)$ and for $\underline{\mathcal{Q}}_2^{(H)}(\lambda, \nu)$ for $H < J-2$, which will be used later on, see eqs. 10.13,^{121,122}
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$$\begin{aligned} (\mathcal{Q}_2^{(H)}(\lambda, \nu))^{\frac{1}{2}} &\leq C^{*(1)} \frac{(H+2)!}{(H+2)^{\alpha}} \frac{e^{H(\delta+\Gamma(\lambda))}}{\rho_{0,1}^{(H+2)}} \\ (\underline{\mathcal{Q}}_2^{(H)}(\lambda, \nu))^{\frac{1}{2}} &\leq C^{*(1)} \frac{(H+2)!}{(H+2)^{\alpha}} \frac{e^{H(\delta+\Gamma(\lambda))}}{\rho_{0,1}^{(H+2)}}. \end{aligned} \quad (15.53)$$

We write

$$\begin{aligned} \mathcal{Q}_2^{(J-2)}(\lambda, \nu) + \underline{\mathcal{Q}}_2^{(J-2)}(\lambda, \nu) &\leq \mathcal{Q}_{(0),2}^{(J-2)}(\lambda, \nu) \\ &+ \left(\sup_{\lambda' \in [\lambda_0, \lambda]} \mathcal{Q}_2^{(J-2)}(\lambda', \nu) \right)^{\frac{1}{2}} \left\{ \sum_{k=0}^{J-2} \left[\int_{\lambda_0}^{\lambda} d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^k)^{(O)} \pi_{\rho}^{\tau} \hat{\mathcal{L}}_O^k \mathbf{D}(\tilde{W}^{(J-1-k)})|^2 \right)^{\frac{1}{2}} \right] \right. \\ &\left. + \sum_{k=0}^{J-2} \left[\sum_{l=1}^k \binom{k}{l} \int_{\lambda_0}^{\lambda} d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^l)^{(O)} \pi_{\rho}^{\tau} \hat{\mathcal{L}}_O^{k-l} \mathbf{D}(\tilde{W}^{(J-1-k)})|^2 \right)^{\frac{1}{2}} \right] \right\} \end{aligned} \quad (15.54)$$

Let us consider the first sum in the curled brackets, first of all we write

$$\begin{aligned} &\left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^k)^{(O)} \pi_{\rho}^{\tau} \hat{\mathcal{L}}_O^k \mathbf{D}(\tilde{W}^{(J-1-k)})|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sup_{V(\lambda, \nu)} |r^{\sigma(O)} \pi|^2 \right)^{\frac{1}{2}} \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 r^{-2\sigma} |\hat{\mathcal{L}}_O^k \mathbf{D} \hat{\mathcal{L}}_O^{(J-1-k)} W|^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (15.55)$$

We assume (easy to prove¹²⁴) that we can move \mathbf{D} on the left paying a price of

¹²¹ The consistency follows recalling that $\mathcal{Q}_2^{(H)}$ allows to estimate $|\mathcal{L}_O W^{(H)}|_{p,S}$.

¹²² Notice that, as the constant $C^{(1)}$ bounds the sum of the first H norms we should choose a different, smaller constant. We call it still $C^{(1)}$ for the sake of simplicity.

¹²³ We have to remember that we are proving the estimate for the $\mathcal{Q}^{(H)}$ in an inductive way which implies that these estimates (we expect correct) have been already proved for $H < J-2$ and we prove now for $H = J-2$.

¹²⁴ Notice, in fact, that an extra factor $\leq O(J)$ is produced for each term when we commute \mathbf{D} with the \mathcal{L}_O , implying we have also a term to estimate multiplied by a factor $\leq O(J^2)$, but there is one $\hat{\mathcal{L}}_O$ less which allows to control this extra $O(J)$ factor. Therefore this terms are controlled as the more delicate one with the difference the we can use the inductive assumption on the $\mathcal{Q}^{(J-2)}$.

harmless lower order terms. Moreover the weight r^σ is not specified as it depends on which component of $^{(O)}\pi$ one is considering, but it will turn out that it is systematically the appropriate one. Observe that, see for instance [Ch-Kl] eq. (7.5.12a), we can expect the following estimate we have to investigate carefully later on,

$$\begin{aligned}
& \int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 r^{-2\sigma} |\hat{\mathcal{L}}_O^k \mathbf{D} \tilde{W}^{(J-1-k)}|^2 \leq \int_{\nu_0}^\nu \tau_+^4 r^{-2\sigma} \int_{S(\lambda', \nu)} \frac{1}{r^2(\lambda', \nu')} |\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)}|^2 \\
& \leq \frac{1}{r^{2+2\sigma}(\lambda', \nu_0)} \int_{\nu_0}^\nu \tau_+^4 \int_{S(\lambda', \nu)} |\hat{\mathcal{L}}_O^2 \Psi(\tilde{W}^{(J-2)})|^2 \leq \frac{1}{r^{2+2\sigma}(\lambda', \nu_0)} \mathcal{Q}_2^{(J-2)}(\lambda', \nu) \\
& \leq \frac{c}{|\lambda'|^{2+2\sigma}} \mathcal{Q}_2^{(J-2)}(\lambda', \nu) .
\end{aligned} \tag{15.56}$$

and from it, observing that

$$\left(\sup_{\lambda' \in [\lambda_0, \lambda]} |r^\sigma {}^{(O)}\pi|^2 \right)^{\frac{1}{2}} = O(\varepsilon) ,$$

$$\begin{aligned}
& \sum_{k=0}^{J-2} \left[\int_{\lambda_0}^\lambda d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |({}^{(O)}\pi^\tau)_\rho \hat{\mathcal{L}}_O^k \mathbf{D}(\tilde{W}^{(J-1-k)})|^2 \right)^{\frac{1}{2}} \right] \\
& \leq c \left(\sup_{V(\lambda, \nu)} |r^\sigma {}^{(O)}\pi|^2 \right)^{\frac{1}{2}} (J-1) \int_{\lambda_0}^\lambda d\lambda' \frac{1}{|\lambda'|^{1+\sigma}} \left(\mathcal{Q}_2^{(J-2)}(\lambda', \nu) \right)^{\frac{1}{2}} \tag{15.57} \\
& \leq c\epsilon_0 (J-1) \int_{\lambda_0}^\lambda d\lambda' \frac{1}{|\lambda'|^{1+\sigma}} \left(\mathcal{Q}_2^{(J-2)}(\lambda', \nu) \right)^{\frac{1}{2}} ,
\end{aligned}$$

so that, we have,

$$\mathcal{Q}_2^{(J-2)}(\lambda, \nu) + \underline{\mathcal{Q}}_2^{(J-2)}(\lambda, \nu) \leq \mathcal{Q}_{(0),2}^{(J-2)}(\lambda, \nu) \tag{15.58}$$

$$\begin{aligned}
& + \left(\sup_{\lambda' \in [\lambda_0, \lambda]} \mathcal{Q}_2^{(J-2)}(\lambda', \nu) \right)^{\frac{1}{2}} \left\{ c\epsilon(J-1) \int_{\lambda_0}^\lambda d\lambda' \frac{1}{|\lambda'|^{1+\sigma}} \left(\mathcal{Q}_2^{(J-2)}(\lambda', \nu) \right)^{\frac{1}{2}} \right. \\
& + \sum_{k=0}^{J-2} \left[\sum_{l=1}^k \binom{k}{l} \int_{\lambda_0}^\lambda d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^l {}^{(O)}\pi^\tau)_\rho \hat{\mathcal{L}}_O^{k-l} \mathbf{D}(\tilde{W}^{(J-1-k)})|^2 \right)^{\frac{1}{2}} \right] \Bigg\} \\
& .
\end{aligned} \tag{15.59}$$

Remark:

Let us forget for a moment the second sum in the curly brackets and observe that many terms of the error can be controlled via the inductive estimates, but this cannot be done for the first part as, due to the factor $\epsilon(J-1)$, this term cannot be written as $\mathcal{Q}_{1,2}^{(J-2)}$ multiplied by a small factor. Therefore we need an

internal bootstrap mechanism: we prove that the estimate 15.53 for $\mathcal{Q}_{1,2}^{(H=J-2)}$ is satisfied in a small region starting from the the initial data surface $\underline{C}_0 \cup \underline{C}_0$ going up in ν from ν_0 to $\nu_0 + \delta$ and in λ from λ_0 to $\lambda_0 + \delta$ then we assume that the largest region where this estimate holds with the constant $C^{(1)}$ be the one with $\nu \in [\nu_0, \bar{\nu}]$ and $\lambda \in [\lambda_0, \bar{\lambda}]$ and prove that this region can be extended going up in ν and up in λ . We do it now considering only the contribution of the first term in the curly brackets, but the procedure can be done in the same way considering all the error terms.

Let us go back to our proof of the boundedness of the \mathcal{Q} norms via the error estimates. As we said the estimates we want to prove are,

$$(\mathcal{Q}_2^{(J-2)}(\lambda, \nu))^{\frac{1}{2}} \leq C^{*(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J}, \quad (15.60)$$

Recalling that we are estimating the second term of inequality 15.59 and that $\Gamma(\lambda)$ is increasing in λ , and using the bootstrap assumption

$$\begin{aligned} & \mathcal{Q}_2^{(J-2)}(\lambda, \nu) + \underline{\mathcal{Q}}_2^{(J-2)}(\lambda, \nu) \leq \mathcal{Q}_{(0),2}^{(J-2)}(\lambda, \nu) \\ & + \left(C^{*(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma_0(\lambda))}}{\rho_{0,1}^J} \right) \left\{ c\epsilon(J-1) \int_{\lambda_0}^{\lambda} d\lambda' \frac{1}{|\lambda'|^{1+\sigma}} \left(\mathcal{Q}_2^{(J-2)}(\lambda', \nu) \right)^{\frac{1}{2}} \right\} \\ & \leq \left(C_0^{(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma_0(\lambda))}}{\rho_{0,1}^J} \right)^2 \\ & + \left(C^{*(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right) \left\{ c\epsilon(J-1) \int_{\lambda_0}^{\lambda} d\lambda' \frac{1}{|\lambda'|^{1+\sigma}} \left(\mathcal{Q}_2^{(J-2)}(\lambda', \nu) \right)^{\frac{1}{2}} \right\} \\ & \leq \left(C_0^{(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma_0(\lambda))}}{\rho_{0,1}^J} \right)^2 \\ & + \left[(c\epsilon) \left(C^{*(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho^J} \right)^2 (J-1) \int_{\lambda_0}^{\lambda} d\lambda' \frac{1}{|\lambda'|^{1+\sigma}} e^{(J-2)(-\Gamma(\lambda)+\Gamma(\lambda'))} \right] \\ & \leq \left(C_0^{(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma_0(\lambda))}}{\rho_{0,1}^J} \right)^2 + \left[(c\epsilon) \left(C^{*(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right)^2 (J-1) \int_{\lambda_0}^{\lambda} d\lambda' \frac{1}{|\lambda'|^2} e^{(J-2)\frac{(\lambda'-\lambda)}{\lambda'\lambda}} \right]. \end{aligned} \quad (15.61)$$

Therefore

$$\begin{aligned} & \left(\mathcal{Q}_2^{(J-2)}(\lambda, \nu) + \underline{\mathcal{Q}}_2^{(J-2)}(\lambda, \nu) \right)^{\frac{1}{2}} \\ & \leq \left(C_0^{(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma_0(\lambda))}}{\rho_{0,1}^J} \right) + (c\epsilon) \left(C^{*(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right) \left[(J-1) \int_{\lambda_0}^{\lambda} d\lambda' \frac{1}{|\lambda'|^2} e^{(J-2)\frac{(\lambda'-\lambda)}{\lambda'\lambda}} \right]^{\frac{1}{2}} \end{aligned} \quad (15.62)$$

Observe that denoting,

$$x = \frac{(\lambda' - \lambda)}{\lambda' \lambda} \quad ; \quad \frac{dx}{d\lambda'} = \frac{1}{\lambda'^2} \quad ,$$

therefore

$$\int_{\lambda_0}^{\lambda} d\lambda' \frac{1}{\lambda'^2} e^{(J-2)\frac{(\lambda'-\lambda)}{\lambda'\lambda}} = - \int_0^{x_0} dx e^{(J-2)x} = \frac{c}{J-2} \quad (15.63)$$

as $x_0 = \frac{(\lambda_0 - \lambda)}{\lambda_0 \lambda} < 0$ and, finally,

$$\begin{aligned} \left(\mathcal{Q}_2^{(J-2)}(\lambda, \nu) + \underline{\mathcal{Q}}_2^{(J-2)}(\lambda, \nu) \right)^{\frac{1}{2}} &\leq \left[C_0^{(1)} + (c'\varepsilon) C^{*(1)} \right] \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta + \Gamma(\lambda))}}{\rho_{0,1}^J} \right) \\ &< \left(C^{*(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta + \Gamma(\lambda))}}{\rho_{0,1}^J} \right) \quad , \end{aligned} \quad (15.64)$$

choosing ε sufficiently small, which implies that it can be extended to any ν up to ν_* and to the largest value of $\lambda = \lambda_1$. Therefore the result holds everywhere and the control of the more delicate term of the error is done.

There is, see later on, another term which requires the same internal bootstrap which therefore has to be done considering all the terms, but now we investigate the second sum in 15.54 which can be estimated via the inductive assumptions, namely

$$\sum_{k=0}^{J-2} \left[\sum_{l=1}^k \binom{k}{l} \int_{\lambda_0}^{\lambda} d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^l \pi_{\rho}^{\tau} \hat{\mathcal{L}}_O^{k-l} \mathbf{D}(\tilde{W}^{(J-1-k)}))|^2 \right)^{\frac{1}{2}} \right] .$$

To control this sum, assuming the estimates 15.53

$$\begin{aligned} (\mathcal{Q}_2^{(H)}(\lambda, \nu))^{\frac{1}{2}} &\leq C^{*(1)} \frac{(H+2)!}{(H+2)^\alpha} \frac{e^{H(\delta + \Gamma(\lambda))}}{\rho_{0,1}^{(H+2)}} \\ (\underline{\mathcal{Q}}_2^{(H)}(\lambda, \nu))^{\frac{1}{2}} &\leq C^{*(1)} \frac{(H+2)!}{(H+2)^\alpha} \frac{e^{H(\delta + \Gamma(\lambda))}}{\rho_{0,1}^{(H+2)}} \end{aligned}$$

and the following estimate,

$$|r^{\sigma - \frac{2}{p}} \hat{\mathcal{L}}_O^l \pi_{p, S(\lambda', \nu')}^{(O)}| \leq c_1 \frac{l!}{l^\alpha} \frac{e^{(l-2)(\delta + \Gamma(\lambda'))}}{\rho_{0,1}^l} \quad , \quad (15.65)$$

we have, assuming for the moment $k \leq J/2$,

$$\left[\sum_{l=1}^k \binom{k}{l} \int_{\lambda_0}^{\lambda} d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^l \pi_{\rho_{0,1}}^{\tau} \hat{\mathcal{L}}_O^{k-l} \mathbf{D}(\tilde{W}^{(J-1-k)}))|^2 \right)^{\frac{1}{2}} \right]$$

$$\begin{aligned}
&\leq c \sum_{l=1}^k \binom{k}{l} \left(\sup_{V(\bar{\lambda}, \bar{\nu})} |r^\sigma \hat{\mathcal{L}}_O^l ({}^O\pi|^2) \right)^{\frac{1}{2}} \int_{\lambda_0}^{\bar{\lambda}} d\lambda' \frac{1}{|\lambda'|^{2+2\sigma}} \left(\mathcal{Q}_2^{(J-2-l)}(\lambda', \nu) \right)^{\frac{1}{2}} \\
&\leq cc_1 \sum_{l=1}^k \binom{k}{l} \frac{(l+2)!}{(l+2)^\alpha} \frac{e^{(l-1)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^{l+1}} \int_{\lambda_0}^{\lambda} d\lambda' \frac{1}{|\lambda'|^{2+2\sigma}} C^{(1)} \frac{(J-l)!}{(J-l)^\alpha} \frac{e^{(J-l-2)(\delta+\Gamma(\lambda'))}}{\rho_{0,1}^{(J-l)}} \\
&\leq cc_1 C^{(1)} \sum_{l=1}^k \binom{k}{l} \frac{(l+2)!}{(l+2)^\alpha} \frac{e^{(l-1)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^{l+1}} \int_{\lambda_0}^{\lambda} d\lambda' \frac{1}{|\lambda'|^{2+2\sigma}} \frac{(J-l)!}{(J-l)^\alpha} \frac{e^{(J-l-2)(\delta+\Gamma(\lambda'))}}{\rho_{0,1}^{(J-l)}} \\
&\leq cc_1 C^{(1)} \sum_{l=1}^k \binom{k}{l} \frac{(l+2)!}{(l+2)^\alpha} \frac{(J-l)!}{(J-l)^\alpha} \frac{e^{(l-1)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^{J+1}} \int_{\lambda_0}^{\lambda} d\lambda' \frac{1}{|\lambda'|^{2+2\sigma}} e^{(J-l-2)(\delta+\Gamma(\lambda'))} \\
&\leq C^{(1)} \left(\frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right) \left(cc_1 \frac{e^{-\delta}}{\rho_{0,1}} \right) \sum_{l=1}^k \binom{k}{l} \frac{(l+2)!}{(l+2)^\alpha} \frac{(J-l)!}{(J-l)^\alpha} \int_{\lambda_0}^{\lambda} d\lambda' \frac{1}{|\lambda'|^{2+2\sigma}} e^{-(J-l-2)(\Gamma(\lambda)-\Gamma(\lambda'))} .
\end{aligned}$$

Denoting,

$$x = \frac{(\lambda' - \lambda)}{\lambda' \lambda} \quad ; \quad \frac{dx}{d\lambda'} = \frac{1}{\lambda'^2} ,$$

we have

$$\begin{aligned}
&\int_{\lambda_0}^{\lambda} d\lambda' \frac{1}{|\lambda'|^{2+2\sigma}} e^{-(J-l-2)(\Gamma(\lambda)-\Gamma(\lambda'))} \leq \int_{\lambda_0}^{\lambda} d\lambda' \frac{1}{|\lambda'|^2} e^{(J-2-l)\frac{(\lambda'-\lambda)}{\lambda'\lambda}} \\
&= - \int_0^{x_0} dx e^{(J-2-l)x} = \frac{c'}{(J-2-l)} .
\end{aligned} \tag{15.66}$$

Therefore

$$\begin{aligned}
&\sum_{k=0}^{\lfloor \frac{J}{2} \rfloor} \left[\sum_{l=1}^k \binom{k}{l} \int_{\lambda_0}^{\lambda} d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^l ({}^O\pi)^\tau)_\rho \hat{\mathcal{L}}_O^{k-l} \mathbf{D}(\tilde{W}^{(J-1-k)})|^2 \right)^{\frac{1}{2}} \right] \\
&\leq C^{(1)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right) \left(cc' c_1 \frac{e^{-\delta}}{\rho_{0,1}} \right) \left\{ \sum_{k=0}^{\lfloor \frac{J}{2} \rfloor} \sum_{l=1}^k \binom{k}{l} \frac{(l+2)!}{(l+2)^\alpha} \frac{(J-l)!}{(J-l)^\alpha} \frac{J^\alpha}{J!} \frac{1}{(J-2-l)} \right\} .
\end{aligned}$$

Observe that

$$\begin{aligned}
\{\cdot \dots \cdot\} &= \sum_{k=0}^{\lfloor \frac{J}{2} \rfloor} \sum_{l=1}^k \frac{k!}{(k-l)!l!} \frac{(l+2)!}{(l+2)^\alpha} \frac{(J-l)!}{(J-l)^\alpha} \frac{J^\alpha}{J!} \frac{1}{(J-2-l)} \\
&= \sum_{k=0}^{\lfloor \frac{J}{2} \rfloor} \sum_{l=1}^k \frac{1}{(l+2)^\alpha} \frac{(l+2)!}{l!} \left[\frac{k!}{(k-l)!} \frac{(J-l)!}{J!} \frac{J^\alpha}{(J-l)^\alpha} \frac{1}{(J-2-l)} \right] \\
&\leq \frac{1}{J} \sum_{k=0}^{\lfloor \frac{J}{2} \rfloor} \sum_{l=1}^k \frac{1}{(l+2)^\alpha} \frac{(l+2)!}{l!} \left[\frac{k!}{(k-l)!} \frac{(J-l)!}{J!} \frac{J^\alpha}{(J-l)^\alpha} \frac{J}{(J-2-l)} \right]
\end{aligned} \tag{15.67}$$

$$\begin{aligned}
&\leq \frac{1}{J} \sum_{k=0}^{\lfloor \frac{J}{2} \rfloor} \sum_{l=1}^k \frac{1}{(l+2)^\alpha} \frac{(l+2)!}{l!} \left[2 \frac{k!}{(k-l)!} \frac{(J-l)!}{J!} 2^\alpha \right] \\
&\leq c \left(\frac{2^{\alpha+1}}{J} \right) \sum_{k=0}^{\lfloor \frac{J}{2} \rfloor} \left[\sum_{l=1}^k \frac{1}{(l+2)^{\alpha-2}} \right] \leq c_2
\end{aligned}$$

choosing $\alpha > 3$. Therefore finally

$$\begin{aligned}
&\sum_{k=0}^{\lfloor \frac{J}{2} \rfloor} \left[\sum_{l=1}^k \binom{k}{l} \int_{\lambda_0}^\lambda d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^l \pi_\rho^{(O)})^\tau \hat{\mathcal{L}}_O^{k-l} \mathbf{D}(\tilde{W}^{(J-1-k)})|^2 \right)^{\frac{1}{2}} \right] \\
&\leq C^{(1)} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0 + \Xi_0(\lambda))}}{\rho_{0,1}^J} \right) \left(cc' c_1 c_2 \frac{e^{-\delta}}{\rho_{0,1}} \right). \quad (15.68)
\end{aligned}$$

Let us investigate the same sum with $k \geq J/2$. We have to divide it in two sums, the first with $l \leq J/2$ and the second with $l > J/2$. The first sum is treated exactly as the previous one so we are left with estimating,

$$\sum_{k=\lfloor \frac{J}{2} \rfloor + 1}^{J-2} \sum_{l=\lfloor \frac{J}{2} \rfloor + 1}^k \binom{k}{l} \int_{\lambda_0}^\lambda d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^l \pi_\rho^{(O)})^\tau \hat{\mathcal{L}}_O^{k-l} \mathbf{D}(\tilde{W}^{(J-1-k)})|^2 \right)^{\frac{1}{2}}.$$

We write

$$\begin{aligned}
&\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^l \pi_\rho^{(O)})^\tau \hat{\mathcal{L}}_O^{k-l} \mathbf{D}(\tilde{W}^{(J-1-k)})|^2 \\
&= \int_{\nu_0}^\nu d\nu' \tau_+^4 \int_{S(\lambda', \nu')} d\mu |\mathcal{L}_O^l \pi_\rho^{(O)}|^\tau |\hat{\mathcal{L}}_O^{k-l} \mathbf{D}(\tilde{W}^{(J-1-k)})|^2 \\
&\leq \int_{\nu_0}^\nu d\nu' \tau_+^4 \left(\int_{S(\lambda', \nu')} d\mu |\mathcal{L}_O^l \pi_\rho^{(O)}|^\tau \right)^{\frac{1}{2}} \left(\int_{S(\lambda', \nu')} d\mu |\hat{\mathcal{L}}_O^{k-l} \mathbf{D}(\tilde{W}^{(J-1-k)})|^4 \right)^{\frac{1}{2}} \\
&\leq \int_{\nu_0}^\nu d\nu' \tau_+^4 |\mathcal{L}_O^l \pi_\rho^{(O)}|_{4, S(\lambda', \nu')}^\tau |\hat{\mathcal{L}}_O^{k-l} \mathbf{D}(\tilde{W}^{(J-1-k)})|_{4, S(\lambda', \nu')}^2. \quad (15.69)
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{k=\lfloor \frac{J}{2} \rfloor + 1}^{J-2} \sum_{l=\lfloor \frac{J}{2} \rfloor + 1}^k \binom{k}{l} \int_{\lambda_0}^\lambda d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^l \pi_\rho^{(O)})^\tau \hat{\mathcal{L}}_O^{k-l} \mathbf{D}(\tilde{W}^{(J-1-k)})|^2 \right)^{\frac{1}{2}} \quad (15.70) \\
&\leq \sum_{k=\lfloor \frac{J}{2} \rfloor + 1}^{J-2} \sum_{l=\lfloor \frac{J}{2} \rfloor + 1}^k \binom{k}{l} \int_{\lambda_0}^\lambda d\lambda' \left(\int_{\nu_0}^\nu d\nu' \tau_+^4 |\mathcal{L}_O^l \pi_\rho^{(O)}|_{4, S(\lambda', \nu')}^\tau |\hat{\mathcal{L}}_O^{k-l} \mathbf{D}(\tilde{W}^{(J-1-k)})|_{4, S(\lambda', \nu')}^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

To estimate 15.70 we have to make an assumption about the estimate of $|r^\sigma \hat{\mathcal{L}}_O^{(O)} \pi|^2$. As $^{(O)}\pi$ has to be treated as a connection coefficient we can assume the following estimates coherent with the inductive assumptions for the Lie derivatives of the connection coefficients, see later on.

$$|r^{\sigma - \frac{2}{p}} \mathcal{L}_O^l {}^{(O)}\pi|_{p, S(\lambda', \nu')} \leq c_1 \frac{l!}{l^\alpha} \frac{e^{(l-2)(\delta + \Gamma(\lambda'))}}{\rho_{0,1}^l} \quad (15.71)$$

with $\sigma \geq 2$ if $l > 0$, $= 1$ if $l = 0$ and $c_1 = O(\epsilon_0)$ if $l \leq 7$. Moreover

$$\begin{aligned} |\hat{\mathcal{L}}_O^{k-l} \mathbf{D}(\tilde{W}^{(J-1-k)})|_{4, S(\lambda', \nu')} &\leq |r^{-1} \Psi(\tilde{W}^{(J-l)})|_{4, S(\lambda', \nu')} \\ &\leq r^{-1-(1-\frac{2}{4})} |\lambda'|^{-\frac{5}{2}} \left(C^{(1)} \frac{(J-l)!}{(J-l)^\alpha} \frac{e^{(J-l-1)(\delta + \Gamma(\lambda'))}}{\rho_{0,1}^{J-l+1}} \right) \end{aligned} \quad (15.72)$$

Therefore

$$\begin{aligned} &\sum_{k=\lfloor \frac{J}{2} \rfloor + 1}^{J-2} \sum_{l=\lfloor \frac{J}{2} \rfloor + 1}^k \binom{k}{l} \int_{\lambda_0}^\lambda d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^l {}^{(O)}\pi)_\rho^\tau| \hat{\mathcal{L}}_O^{k-l} \mathbf{D}(\tilde{W}^{(J-1-k)})|^2 \right)^{\frac{1}{2}} \quad (15.73) \\ &\leq \sum_{k=\lfloor \frac{J}{2} \rfloor + 1}^{J-2} \sum_{l=\lfloor \frac{J}{2} \rfloor + 1}^k \binom{k}{l} \int_{\lambda_0}^\lambda d\lambda' \left(\int_{\nu_0}^\nu d\nu' \tau_+^4 |\mathcal{L}_O^l {}^{(O)}\pi_\rho^\tau|_{4, S(\lambda', \nu')}^2 |\hat{\mathcal{L}}_O^{k-l} \mathbf{D}(\tilde{W}^{(J-1-k)})|_{4, S(\lambda', \nu')}^2 \right)^{\frac{1}{2}} \\ &\leq \sum_{k=\lfloor \frac{J}{2} \rfloor + 1}^{J-2} \sum_{l=\lfloor \frac{J}{2} \rfloor + 1}^k \binom{k}{l} \left(c_1 \frac{l!}{l^\alpha} \frac{1}{\rho_{0,1}^l} \right) \left(C^{(1)} \frac{(J-l)!}{(J-l)^\alpha} \frac{1}{\rho_{0,1}^{J-l+1}} \right) \\ &\quad \cdot \int_{\lambda_0}^\lambda d\lambda' \left(\int_{\nu_0}^\nu d\nu' \tau_+^4 \left(\frac{1}{r^3(\lambda', \nu')} e^{2(l-2)\Gamma(\lambda')} \right) \left(\frac{1}{|\lambda'|^5 r^3} e^{2(J-l-1)\Gamma(\lambda')} \right) \right)^{\frac{1}{2}} \\ &\leq \sum_{k=\lfloor \frac{J}{2} \rfloor + 1}^{J-2} \sum_{l=\lfloor \frac{J}{2} \rfloor + 1}^k \binom{k}{l} \left(c_1 C^{(1)} \frac{l!}{l^\alpha} \frac{(J-l)!}{(J-l)^\alpha} \frac{e^{(J-3)\delta}}{\rho_{0,1}^{J+1}} \right) \int_{\lambda_0}^\lambda d\lambda' \frac{1}{|\lambda'|^{\frac{5}{2}}} e^{(J-3)\Gamma(\lambda')} \left(\int_{\nu_0}^\nu d\nu' \frac{1}{\nu'^2} \right)^{\frac{1}{2}} \\ &\leq \left(C^{(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta + \Gamma(\lambda, \nu))}}{\rho_{0,1}^J} \right) \left(c_1 \frac{e^{-\delta}}{\rho_{0,1}} \right) \left[\sum_{k=\lfloor \frac{J}{2} \rfloor + 1}^{J-2} \sum_{l=\lfloor \frac{J}{2} \rfloor + 1}^k \binom{k}{l} \frac{J^\alpha}{J!} \frac{l!}{l^\alpha} \frac{(J-l)!}{(J-l)^\alpha} \right] \\ &\quad \cdot \left(\int_{\lambda_0}^\lambda \frac{d\lambda'}{|\lambda'|^{\frac{5}{2}}} e^{-(J-2)(\Gamma(\lambda) - \Gamma(\lambda'))} e^{-\Gamma(\lambda')} \right) \\ &\leq \left(C^{(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta + \Gamma(\lambda, \nu))}}{\rho_{0,1}^J} \right) \left(c_1 \frac{e^{-\delta} e^{\frac{1}{\nu_0}}}{\rho_{0,1}} \right) \left[\sum_{k=\lfloor \frac{J}{2} \rfloor + 1}^{J-2} \sum_{l=\lfloor \frac{J}{2} \rfloor + 1}^k \binom{k}{l} \frac{J^\alpha}{J!} \frac{l!}{l^\alpha} \frac{(J-l)!}{(J-l)^\alpha} \right] \\ &\quad \cdot \left(\int_{\lambda_0}^\lambda \frac{d\lambda'}{|\lambda'|^{\frac{5}{2}}} e^{-(J-2)(\Gamma(\lambda) - \Gamma(\lambda'))} \right), \end{aligned}$$

the last inequality as $e^{-\Gamma(\lambda')} < 1$. Observe that, for suitable c''

$$\int_{\lambda_0}^{\lambda} \frac{d\lambda'}{|\lambda'|^{\frac{5}{2}}} e^{(J-2)(-\Gamma(\lambda)-\Gamma(\lambda'))} = - \int_0^{x_0} dx x^{\frac{1}{2}} e^{(J-2)x} \leq \frac{c'}{(J-2)^{\frac{3}{2}}} \quad (15.74)$$

and substituting in the previous estimate we obtain

$$\begin{aligned} & \sum_{k=\lfloor \frac{J}{2} \rfloor + 1}^{J-2} \sum_{l=\lfloor \frac{J}{2} \rfloor + 1}^k \binom{k}{l} \int_{\lambda_0}^{\lambda} d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^l)^{(O)} \pi_{\rho}^{\tau} \hat{\mathcal{L}}_O^{k-l} \mathbf{D}(\tilde{W}^{(J-1-k)})|^2 \right)^{\frac{1}{2}} \\ & \leq \left(C^{(1)} \frac{J!}{J^{\alpha}} \frac{e^{(J-2)(\delta+\Gamma(\lambda, \nu))}}{\rho_{0,1}^J} \right) \left(c_1 c' \frac{e^{-\delta} e^{\frac{1}{\nu_0}}}{\rho_{0,1}} \right) \left[\sum_{k=\lfloor \frac{J}{2} \rfloor + 1}^{J-2} \sum_{l=\lfloor \frac{J}{2} \rfloor + 1}^k \binom{k}{l} \frac{J^{\alpha}}{J!} \frac{l!}{l^{\alpha}} \frac{(J-l)!}{(J-l)^{\alpha}} \frac{1}{(J-2)^{\frac{3}{2}}} \right]. \end{aligned} \quad (15.75)$$

We have to prove that the factor $[\cdot \cdot]$ is bounded by a constant c . Choosing δ sufficiently large this term is under control. In fact

$$\begin{aligned} & \left[\sum_{k=\lfloor \frac{J}{2} \rfloor + 1}^{J-2} \sum_{l=\lfloor \frac{J}{2} \rfloor + 1}^k \binom{k}{l} \frac{J^{\alpha}}{J!} \frac{l!}{l^{\alpha}} \frac{(J-l)!}{(J-l)^{\alpha}} \frac{1}{(J-2)^{\frac{3}{2}}} \right] \\ & = \frac{J^{\alpha}}{J!} \frac{1}{(J-2)^{\frac{3}{2}}} \sum_{k=\lfloor \frac{J}{2} \rfloor + 1}^{J-2} \sum_{l=\lfloor \frac{J}{2} \rfloor + 1}^k \frac{k!}{(k-l)!} \frac{1}{l^{\alpha}} \frac{(J-l)!}{(J-l)^{\alpha}} \\ & = \frac{1}{J!} \frac{1}{(J-2)^{\frac{3}{2}}} \sum_{k=\lfloor \frac{J}{2} \rfloor + 1}^{J-2} \sum_{l=\lfloor \frac{J}{2} \rfloor + 1}^k \frac{k!}{(k-l)!} \frac{J^{\alpha}}{l^{\alpha}} \frac{(J-l)!}{(J-l)^{\alpha}} \\ & \leq \frac{1}{(J-2)^{\frac{3}{2}}} \sum_{k=\lfloor \frac{J}{2} \rfloor + 1}^{J-2} \sum_{l=\lfloor \frac{J}{2} \rfloor + 1}^k \left[\frac{k!(J-l)!}{J!(k-l)!} \right] \left[\frac{J^{\alpha}}{l^{\alpha}} \right] \frac{1}{(J-l)^{\alpha}} \\ & \leq \left(\sup \left[\frac{k!(J-l)!}{J!(k-l)!} \right] \left[\frac{1}{(J-2)^{\frac{3}{2}}} \frac{J^{\alpha}}{l^{\alpha}} \right] \right) \sum_{k=\lfloor \frac{J}{2} \rfloor + 1}^{J-2} \left(\sum_{l=\lfloor \frac{J}{2} \rfloor + 1}^k \frac{1}{(J-l)^{\alpha}} \right) \\ & \leq c \frac{1}{(J-2)^{\frac{3}{2}}} \sum_{k=\lfloor \frac{J}{2} \rfloor + 1}^{J-2} c' \leq \frac{c''}{(J-2)^{\frac{1}{2}}}. \end{aligned}$$

15.4.1 Going back to the error terms:

Recalling the previous expressions, see eq. 15.47 written symbolically, omitting the constants, as

$$(\mathbf{D}^{\mu} \hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\mu\beta\gamma\delta} = \sum_{k=0}^{J-2} \mathcal{L}_O^k \left({}^{(O)}\pi_{\beta}^{\rho} \mathbf{D}^{\mu} (\tilde{W}^{(J-1-k)})_{\mu\rho\gamma\delta} \right) + \sum_{k=0}^{J-1} \mathcal{L}_O^k \left({}^{(O)}\hat{\pi}^{\mu\nu} \mathbf{D}_{\nu} \tilde{W}_{\mu\beta\gamma\delta}^{(J-1-k)} \right)$$

$$+ \sum_{k=0}^{J-1} \mathcal{L}_O^k \left({}^{(O)}\tilde{q}_{\alpha\beta\lambda} \tilde{W}^{(J-1-k)\alpha\lambda}_{\gamma\delta} \right),$$

where

$$\tilde{q}_{\alpha\beta\lambda} = \delta_{\alpha\beta} {}^{(O)}p_\lambda \quad \text{or} \quad {}^{(O)}\tilde{q}_{\alpha\beta\lambda}.$$

We have estimated the contribution of the first sum, the second one should be estimated in the same way, we look here at the contribution of the third one. Therefore we proceed as we had

$$(\mathbf{D}^\mu \hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\mu\beta\gamma\delta} = \sum_{k=0}^{J-1} \mathcal{L}_O^k \left({}^{(O)}\tilde{q}_{\alpha\beta\lambda} \tilde{W}^{(J-1-k)\alpha\lambda}_{\gamma\delta} \right) + [\text{terms under control}].$$

The error, considering only the first sum, has the form

$$\begin{aligned} & \int_{V_{(\lambda,\nu)}} (\mathbf{D}^\mu \hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\mu\rho\gamma\sigma} (\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\beta\delta}^{\rho\sigma} (\bar{K}^\beta \bar{K}^\gamma T^\delta) \\ &= \sum_{k=0}^{J-1} \int_{V_{(\lambda,\nu)}} \mathcal{L}_O^k \left({}^{(O)}\tilde{q}_{\alpha\rho\lambda} \tilde{W}^{(J-1-k)\alpha\lambda}_{\gamma\sigma} \right) (\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\beta\delta}^{\rho\sigma} (\bar{K}^\beta \bar{K}^\gamma T^\delta). \end{aligned} \quad (15.76)$$

Therefore neglecting all the other parts of the error we have to deal with the following inequality,

$$\begin{aligned} & \mathcal{Q}_2^{(J-2)}(\lambda, \nu) + \underline{\mathcal{Q}}_2^{(J-2)}(\lambda, \nu) = \mathcal{Q}_{(0),2}^{(J-2)}(\lambda, \nu) \\ &= \sum_{k=0}^{J-1} \int_{V_{(\lambda,\nu)}} \mathcal{L}_O^k \left({}^{(O)}\tilde{q}_{\alpha\rho\lambda} \tilde{W}^{(J-1-k)\alpha\lambda}_{\gamma\sigma} \right) (\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\beta\delta}^{\rho\sigma} (\bar{K}^\beta \bar{K}^\gamma T^\delta) \\ &= \int_{V_{(\lambda,\nu)}} \mathcal{L}_O^{J-1} \left({}^{(O)}\tilde{q}_{\alpha\rho\lambda} \tilde{W}^{(0)\alpha\lambda}_{\gamma\sigma} \right) (\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\beta\delta}^{\rho\sigma} (\bar{K}^\beta \bar{K}^\gamma T^\delta) \\ &+ \sum_{k=0}^{J-2} \int_{V_{(\lambda,\nu)}} \mathcal{L}_O^k \left({}^{(O)}\tilde{q}_{\alpha\rho\lambda} \tilde{W}^{(J-1-k)\alpha\lambda}_{\gamma\sigma} \right) (\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\beta\delta}^{\rho\sigma} (\bar{K}^\beta \bar{K}^\gamma T^\delta) \end{aligned}$$

where Proceeding as before, we start looking at the first term which is the one we have to add to the “internal bootstrap mechanism”, at its turn it can be decomposed in two terms

$$\begin{aligned} & \int_{V_{(\lambda,\nu)}} \left(\mathcal{L}_O^{J-1} {}^{(O)}\tilde{q}_{\alpha\rho\lambda} \tilde{W}^{(0)\alpha\lambda}_{\gamma\sigma} \right) (\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{\beta\delta}^{\rho\sigma} (\bar{K}^\beta \bar{K}^\gamma T^\delta) \\ &= \int_{V_{(\lambda,\nu)}} \tau_+^4 \left(\mathcal{L}_O^{J-1} {}^{(O)}\tilde{q}_{\alpha\rho\lambda} \tilde{W}^{(0)\alpha\lambda}_{4\sigma} \right) (\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})_{4\delta}^{\rho\sigma} (\bar{K}^\beta \bar{K}^\gamma T^\delta) \\ &= \int_{V_{(\lambda,\nu)}} \tau_+^4 \left(\mathcal{L}_O^{J-1} {}^{(O)}\tilde{q}_{\alpha\cdot\lambda} \tilde{W}^{(0)\alpha\lambda}_{4\cdot} \right) \cdot \alpha(\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)}) \\ &+ \int_{V_{(\lambda,\nu)}} \tau_+^4 \left(\mathcal{L}_O^{J-1} {}^{(O)}\tilde{q}_{\alpha\cdot\lambda} \tilde{W}^{(0)\alpha\lambda}_{4\cdot} \right) \cdot \beta(\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)}). \end{aligned} \quad (15.77)$$

These two terms have the same structure and it is enough to estimate the first one as the same estimate holds for the second,

$$\begin{aligned}
& \int_{V(\lambda, \nu)} \tau_+^4 \left(\mathcal{L}_O^{J-1(O)} \tilde{q}_{\alpha \cdot \lambda} \tilde{W}^{(0)\alpha\lambda}_{\cdot 4} \right) \cdot \alpha(\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)}) \\
&= \sum_{l=0}^{J-1} \binom{J-1}{l} \int_{V(\lambda, \nu)} \tau_+^4 \left((\mathcal{L}_O^l)^{(O)} \tilde{q}_{\alpha \cdot \lambda} \hat{\mathcal{L}}_O^{J-1-l} \tilde{W}^{(0)\alpha\lambda}_{\cdot 4} \right) \cdot \alpha(\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)}) \\
&= \sum_{l=0}^{J-1} \binom{J-1}{l} \int_{\lambda_0}^{\lambda} d\lambda' \left(\int_{C(\lambda'; [\nu, \nu_*])} \tau_+^4 |(\mathcal{L}_O^l)^{(O)} \tilde{q}| \cdot (\hat{\mathcal{L}}_O^{J-1-l} \tilde{W}^{(0)})|^2 \right)^{\frac{1}{2}} \left(\int_{C(\lambda'; [\nu, \nu_*])} \tau_+^4 |\alpha(\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})|^2 \right)^{\frac{1}{2}} \\
&\leq \left(\sup_{\lambda' \in [\lambda_0, \lambda]} \mathcal{Q}_2^{(J-2)}(\lambda', \nu) \right)^{\frac{1}{2}} \left[\sum_{l=0}^{J-1} \binom{J-1}{l} \int_{\lambda_0}^{\lambda} d\lambda' \left(\int_{C(\lambda'; [\nu, \nu_*])} \tau_+^4 |(\mathcal{L}_O^l)^{(O)} \tilde{q}| \cdot (\hat{\mathcal{L}}_O^{J-1-l} \tilde{W}^{(0)})|^2 \right)^{\frac{1}{2}} \right] \\
&\leq \left(\sup_{\lambda' \in [\lambda_0, \lambda]} \mathcal{Q}_2^{(J-2)}(\lambda', \nu) \right)^{\frac{1}{2}} \left[\int_{\lambda_0}^{\lambda} d\lambda' \left(\int_{C(\lambda'; [\nu, \nu_*])} \tau_+^4 |(\mathcal{L}_O^{J-1(O)} \tilde{q}) \cdot (\tilde{W}^{(0)})|^2 \right)^{\frac{1}{2}} \right] \\
&+ \left(\sup_{\lambda' \in [\lambda_0, \lambda]} \mathcal{Q}_2^{(J-2)}(\lambda', \nu) \right)^{\frac{1}{2}} \left[\sum_{l=0}^{J-2} \binom{J-1}{l} \int_{\lambda_0}^{\lambda} d\lambda' \left(\int_{C(\lambda'; [\nu, \nu_*])} \tau_+^4 |(\mathcal{L}_O^l)^{(O)} \tilde{q}| \cdot (\hat{\mathcal{L}}_O^{J-1-l} \tilde{W}^{(0)})|^2 \right)^{\frac{1}{2}} \right]. \quad (15.78)
\end{aligned}$$

Only the first term enters the internal bootstrap mechanism, let us estimate it. Mimicking the previous computations we have

$$\mathcal{Q}_2^{(J-2)}(\lambda, \nu) + \underline{\mathcal{Q}}_2^{(J-2)}(\lambda, \nu) \leq \mathcal{Q}_{(0,2)}^{(J-2)}(\lambda, \nu) \quad (15.79)$$

$$+ \left(C^{*(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta + \Gamma(\lambda))}}{\rho_{0,1}^J} \right) \left\{ \int_{\lambda_0}^{\lambda} d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^{J-1(O)} \tilde{q}) \cdot \tilde{W}^{(0)}|^2 \right)^{\frac{1}{2}} \right\}$$

and, see 15.62

$$\begin{aligned}
& (\mathcal{Q}_2^{(J-2)}(\lambda, \nu) + \underline{\mathcal{Q}}_2^{(J-2)}(\lambda, \nu))^{\frac{1}{2}} \quad (15.80) \\
&\leq \left(C_0^{(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta + \Gamma_0(\lambda))}}{\rho_{0,1}^J} \right) + \int_{\lambda_0}^{\lambda} d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^{J-1(O)} \tilde{q}) \cdot \tilde{W}^{(0)}|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Moreover, proceeding as before,

$$\begin{aligned}
& \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^{J-1(O)} \tilde{q}) \cdot \tilde{W}^{(0)}|^2 \right)^{\frac{1}{2}} = \left(\int_{\nu_0}^{\nu} d\nu' \tau_+^4 \int_{S(\lambda', \nu')} d\nu |(\mathcal{L}_O^{J-1(O)} \tilde{q}) \cdot \tilde{W}^{(0)}|^2 \right)^{\frac{1}{2}} \\
&\leq \left(\int_{\nu_0}^{\nu} d\nu' \tau_+^4 |\mathcal{L}_O^{J-1(O)} \tilde{q}|_{4, S(\lambda', \nu')}^2 |\tilde{W}^{(0)}|_{4, S(\lambda', \nu')}^2 \right)^{\frac{1}{2}} \quad (15.81)
\end{aligned}$$

Using the estimates,¹²⁵

$$|r^{2-\frac{1}{2}} \mathcal{L}_O^{J-1(O)} \tilde{q}|_{4,S(\lambda',\nu')} \leq c_1 \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda'))}}{\rho_{0,1}^J} \quad (15.82)$$

and,¹²⁶

$$|(\tilde{W}^{(0)})|_{4,S(\lambda',\nu')} \leq |\Psi(W)|_{4,S(\lambda',\nu')} \leq r^{-(2-\frac{2}{4})} |\lambda'|^{-\frac{3}{2}} (c\varepsilon) \quad (15.83)$$

we have

$$\begin{aligned} & \left(\int_{C(\lambda';[\nu_0,\nu])} \tau_+^4 |(\mathcal{L}_O^{J-1(O)} \tilde{q}) \cdot \tilde{W}^{(0)}|^2 \right)^{\frac{1}{2}} \leq \left(\int_{\nu_0}^{\nu} d\nu' \tau_+^4 |\mathcal{L}_O^{J-1(O)} \tilde{q}|_{4,S(\lambda',\nu')}^2 |\tilde{W}^{(0)}|_{4,S(\lambda',\nu')}^2 \right)^{\frac{1}{2}} \\ & \leq \frac{1}{|\lambda'|^{\frac{3}{2}}} \left(c_1 \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda'))}}{\rho_{0,1}^J} \right) (c\varepsilon) \left(\int_{\nu_0}^{\nu} d\nu' \frac{\tau_+^4}{r^{3+3}} \right)^{\frac{1}{2}} \\ & \leq \frac{1}{|\lambda'|^{\frac{3}{2}}} \left(c_1 \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda'))}}{\rho_{0,1}^J} \right) (c\varepsilon) \left(\int_{\nu_0}^{\nu} \frac{d\nu'}{\nu'^2} \right)^{\frac{1}{2}} \\ & \leq \frac{c\varepsilon}{|\lambda'|^{\frac{3}{2}}} \left(c_1 \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda'))}}{\rho_{0,1}^J} \right) \end{aligned}$$

and

$$\begin{aligned} & \int_{\lambda_0}^{\lambda} d\lambda' \left(\int_{C(\lambda';[\nu_0,\nu])} \tau_+^4 |(\mathcal{L}_O^{J-1(O)} \tilde{q}) \cdot \tilde{W}^{(0)}|^2 \right)^{\frac{1}{2}} \\ & \leq c\varepsilon \left(c_1 \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda,\nu))}}{\rho_{0,1}^J} \right) \left(\int_{\lambda_0}^{\lambda} \frac{1}{|\lambda'|^{\frac{3}{2}}} \right) \leq c'\varepsilon \left(c_1 \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right). \end{aligned} \quad (15.84)$$

Therefore the terms which have a role in the internal bootstrap are under control and we are left only with terms which can be estimated by the inductive assumptions namely,

$$\begin{aligned} & \sum_{l=0}^{J-2} \binom{J-1}{l} \int_{V(\lambda,\nu)} \tau_+^4 \left((\mathcal{L}_O^l \tilde{q}_{\alpha\cdot\lambda}) \hat{\mathcal{L}}_O^{J-1-l} \tilde{W}^{(0)\alpha\lambda}_{4\cdot} \right) \cdot \alpha(\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)}) \\ & + \sum_{k=0}^{J-2} \int_{V(\lambda,\nu)} \tau_+^4 \mathcal{L}_O^k \left({}^{(O)}\tilde{q}_{\alpha\cdot\lambda} \tilde{W}^{(J-1-k)\alpha\lambda}_{4\cdot} \right) \cdot \alpha(\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)}) \end{aligned}$$

and the analogous term with β . Let us look at the first sum, proceeding as before we have,

$$\sum_{l=0}^{J-2} \binom{J-1}{l} \int_{V(\lambda,\nu)} \tau_+^4 \left((\mathcal{L}_O^l \tilde{q}_{\alpha\cdot\lambda}) \hat{\mathcal{L}}_O^{J-1-l} \tilde{W}^{(0)\alpha\lambda}_{4\cdot} \right) \cdot \alpha(\hat{\mathcal{L}}_O^2 \tilde{W}^{(J-2)})$$

¹²⁵ Again recalling the previous discussion about the estimate for the \mathcal{L}_O^J of the connection coefficients.

¹²⁶ The following estimate is correct due to the index 4 in the beginning of 15.78 which forbids to $W^{(0)}$ of being $\underline{\alpha}(W)$.

$$\leq \left(\sup_{\lambda' \in [\lambda_0, \lambda]} \mathcal{Q}_2^{(J-2)}(\lambda', \nu) \right)^{\frac{1}{2}} \left[\sum_{l=0}^{J-2} \binom{J-1}{l} \int_{\lambda_0}^{\lambda} d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^l \text{}^{(O)} \tilde{q}) \cdot (\hat{\mathcal{L}}_O^{J-1-l} \tilde{W}^{(0)})|^2 \right)^{\frac{1}{2}} \right] \quad (15.85)$$

the second sum can be treated analogously, finally we have

$$\begin{aligned} & (\mathcal{Q}_2^{(J-2)}(\lambda, \nu) + \underline{\mathcal{Q}}_2^{(J-2)}(\lambda, \nu))^{\frac{1}{2}} \leq \\ & \leq \left[\left(C_0^{(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right) + c' \varepsilon \left(c_1 \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right) \right] \\ & + \left[\sum_{l=0}^{J-2} \binom{J-1}{l} \int_{\lambda_0}^{\lambda} d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^l \text{}^{(O)} \tilde{q}) \cdot (\tilde{W}^{(J-1-l)})|^2 \right)^{\frac{1}{2}} \right] \left(\sup_{\lambda' \in [\lambda_0, \lambda]} \mathcal{Q}_2^{(J-2)}(\lambda', \nu) \right)^{\frac{1}{2}} \\ & + \left[\sum_{k=0}^{J-2} \sum_{l=0}^k \binom{k}{l} \int_{\lambda_0}^{\lambda} d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^l \text{}^{(O)} \tilde{q}) \cdot (\tilde{W}^{(J-1-l)})|^2 \right)^{\frac{1}{2}} \right] \left(\sup_{\lambda' \in [\lambda_0, \lambda]} \mathcal{Q}_2^{(J-2)}(\lambda', \nu) \right)^{\frac{1}{2}}. \end{aligned} \quad (15.86)$$

The last two sums can be written in a more compact way as

$$\left[\sum_{k=0}^{J-1} \sum_{l=0}^{k \wedge (J-2)} \binom{k}{l} \int_{\lambda_0}^{\lambda} d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^l \text{}^{(O)} \tilde{q}) \cdot (\tilde{W}^{(J-1-l)})|^2 \right)^{\frac{1}{2}} \right],$$

where $k \wedge (J-2) = \min[k, (J-2)]$.

Therefore assuming for the moment this as the only error contribution we have

$$\begin{aligned} & \mathcal{Q}_2^{(J-2)}(\lambda, \nu) + \underline{\mathcal{Q}}_2^{(J-2)}(\lambda, \nu) \leq \left[\left(C_0^{(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda, \nu))}}{\rho_{0,1}^J} \right) + c' \varepsilon \left(c_1 \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda, \nu))}}{\rho_{0,1}^J} \right) \right] \\ & + \left(\sup_{\lambda' \in [\lambda_0, \lambda]} \mathcal{Q}_2^{(J-2)}(\lambda', \nu) \right)^{\frac{1}{2}} \sum_{k=0}^{k \wedge (J-2)} \left[\sum_{l=0}^k \binom{k}{l} \int_{\lambda_0}^{\lambda} d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^l \text{}^{(O)} \tilde{q}) \cdot (\tilde{W}^{(J-1-l)})|^2 \right)^{\frac{1}{2}} \right]. \end{aligned}$$

We write

$$\begin{aligned} & \mathcal{Q}_2^{(J-2)}(\lambda, \nu) + \underline{\mathcal{Q}}_2^{(J-2)}(\lambda, \nu) \leq \left[\left(C_0^{(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda, \nu))}}{\rho_{0,1}^J} \right) + c' \varepsilon \left(c_1 \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda, \nu))}}{\rho_{0,1}^J} \right) \right] \\ & + \left(\sup_{\lambda' \in [\lambda_0, \lambda]} \mathcal{Q}_2^{(J-2)}(\lambda', \nu) \right)^{\frac{1}{2}} \left\{ \sum_{k=0}^{J-2} \left[\int_{\lambda_0}^{\lambda} d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\text{}^{(O)} \tilde{q}) \cdot (\tilde{W}^{(J-1)})|^2 \right)^{\frac{1}{2}} \right] \right. \\ & \left. + \sum_{k=0}^{k \wedge (J-2)} \left[\sum_{l=1}^k \binom{k}{l} \int_{\lambda_0}^{\lambda} d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^l \text{}^{(O)} \tilde{q}) \cdot (\tilde{W}^{(J-1-l)})|^2 \right)^{\frac{1}{2}} \right] \right\} \end{aligned} \quad (15.87)$$

Let us consider the first sum in the curly brackets, first of all we write

$$\begin{aligned} & \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |({}^{(O)}\tilde{q})(\tilde{W}^{(J-1)})|^2 \right)^{\frac{1}{2}} \\ & \leq \left(\sup_{V(\lambda, \nu)} |r^\sigma {}^{(O)}\tilde{q}|^2 \right)^{\frac{1}{2}} \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 r^{-2\sigma} |\hat{\mathcal{L}}_O^{(J-1)} W|^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (15.88)$$

As before, the weight r^σ is not specified as it depends on which component of ${}^{(O)}\pi$ one is considering, but it will turn out that it is systematically the appropriate one, hence we obtain the following estimate,

$$\begin{aligned} & \int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 r^{-2\sigma} |\tilde{W}^{(J-1)}|^2 \leq \int_{\nu_0}^\nu \tau_+^4 r^{-2\sigma} \int_{S(\lambda', \nu)} \frac{1}{r^2(\lambda', \nu')} |\tilde{W}^{(J-1)}|^2 \\ & \leq \frac{1}{r^{2+2\sigma}(\lambda', \nu_0)} \int_{\nu_0}^\nu \tau_+^4 \int_{S(\lambda', \nu)} |\Psi(\tilde{W}^{(J-1)})|^2 \leq \frac{1}{r^{2+2\sigma}(\lambda', \nu_0)} \mathcal{Q}_2^{(J-3)}(\lambda', \nu) \\ & \leq \frac{c}{|\lambda'|^{2+2\sigma}} \mathcal{Q}_2^{(J-2)}(\lambda', \nu). \end{aligned} \quad (15.89)$$

and from it, observing that

$$\left(\sup_{\lambda' \in [\lambda_0, \lambda]} |r^\sigma {}^{(O)}\tilde{q}|^2 \right)^{\frac{1}{2}} = O(\varepsilon),$$

$$\begin{aligned} & \sum_{k=0}^{J-2} \left[\int_{\lambda_0}^\lambda d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |({}^{(O)}\tilde{q})(\tilde{W}^{(J-1)})|^2 \right)^{\frac{1}{2}} \right] \\ & \leq c \left(\sup_{V(\lambda, \nu)} |r^\sigma {}^{(O)}\tilde{q}|^2 \right)^{\frac{1}{2}} (J-1) \int_{\lambda_0}^\lambda d\lambda' \frac{1}{|\lambda'|^{1+1\sigma}} \left(\mathcal{Q}_2^{(J-3)}(\lambda', \nu) \right)^{\frac{1}{2}} \quad (15.90) \\ & \leq c\epsilon_0 (J-1) \int_{\lambda_0}^\lambda d\lambda' \frac{1}{|\lambda'|^{1+1\sigma}} \left(\mathcal{Q}_2^{(J-3)}(\lambda', \nu) \right)^{\frac{1}{2}}, \end{aligned}$$

so that, we have,

$$\begin{aligned} & \mathcal{Q}_2^{(J-2)}(\lambda, \nu) + \underline{\mathcal{Q}}_2^{(J-2)}(\lambda, \nu) \leq \left[\left(C_0^{(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda, \nu))}}{\rho_{0,1}^J} \right) + c'\varepsilon \left(c_1 \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda, \nu))}}{\rho_{0,1}^J} \right) \right] \\ & + \left(\sup_{\lambda' \in [\lambda_0, \lambda]} \mathcal{Q}_2^{(J-2)}(\lambda', \nu) \right)^{\frac{1}{2}} \left\{ c\epsilon_0 (J-1) \int_{\lambda_0}^\lambda d\lambda' \frac{1}{|\lambda'|^{1+\sigma}} \left(\mathcal{Q}_2^{(J-3)}(\lambda', \nu) \right)^{\frac{1}{2}} \right. \\ & + \left. \sum_{k=0}^{k \wedge (J-2)} \left[\sum_{l=1}^k \binom{k}{l} \int_{\lambda_0}^\lambda d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^l {}^{(O)}\tilde{q}) \hat{\mathcal{L}}_O^l (\tilde{W}^{(J-1-l)})|^2 \right)^{\frac{1}{2}} \right] \right\} \end{aligned} \quad (15.91)$$

1) Let us estimate the first part,

$$\begin{aligned}
& \left(\sup_{\lambda' \in [\lambda_0, \lambda]} \mathcal{Q}_2^{(J-2)}(\lambda', \nu) \right)^{\frac{1}{2}} \left\{ c\epsilon_0(J-1) \int_{\lambda_0}^{\lambda} d\lambda' \frac{1}{|\lambda'|^{1+\sigma}} \left(\mathcal{Q}_2^{(J-3)}(\lambda', \nu) \right)^{\frac{1}{2}} \right. \\
& \leq \left(C^{*(1)} \frac{J!}{J^\alpha} \frac{e^{(J-3)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right) \left\{ c\epsilon_0(J-1) \int_{\lambda_0}^{\lambda} d\lambda' \frac{1}{|\lambda'|^{1+\sigma}} \left(\mathcal{Q}_2^{(J-3)}(\lambda', \nu) \right)^{\frac{1}{2}} \right\} \\
& \leq \left[(c\varepsilon) \left(C^{*(1)} \frac{\rho}{e^{(\delta+\Gamma_0(\lambda))}} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho^J} \right)^2 \frac{(J-1)}{J} \int_{\lambda_0}^{\lambda} d\lambda' \frac{1}{|\lambda'|^{1+\sigma}} e^{(J-2)(-\Gamma(\lambda)+\Gamma(\lambda'))} \right] \\
& \leq \left[(c\varepsilon) \left(C^{*(1)} \frac{\rho}{e^{(\delta+\Gamma_0(\lambda))}} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right)^2 \frac{(J-1)}{J} \int_{\lambda_0}^{\lambda} d\lambda' \frac{1}{|\lambda'|^2} e^{(J-2)\frac{(\lambda'-\lambda)}{\lambda'\lambda}} \right].
\end{aligned}$$

Therefore

$$\begin{aligned}
& \left(\mathcal{Q}_2^{(J-2)}(\lambda, \nu) + \underline{\mathcal{Q}}_2^{(J-2)}(\lambda, \nu) \right)^{\frac{1}{2}} \tag{15.92} \\
& \leq \left(C_0^{(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma_0(\lambda))}}{\rho_{0,1}^J} \right) + (c\varepsilon) \left(C^{*(1)} \frac{\rho}{e^{(\delta+\Gamma_0(\lambda))}} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0+\Gamma_0(\lambda))}}{\rho_{0,1}^J} \right) \left[\frac{(J-1)}{J} \int_{\lambda_0}^{\lambda} d\lambda' \frac{1}{|\lambda'|^2} e^{(J-2)\frac{(\lambda'-\lambda)}{\lambda'\lambda}} \right]^{\frac{1}{2}}
\end{aligned}$$

Observe that denoting,

$$x = \frac{(\lambda' - \lambda)}{\lambda'\lambda} \quad ; \quad \frac{dx}{d\lambda'} = \frac{1}{\lambda'^2},$$

therefore

$$\int_{\lambda_0}^{\lambda} d\lambda' \frac{1}{\lambda'^2} e^{(J-2)\frac{(\lambda'-\lambda)}{\lambda'\lambda}} = - \int_0^{x_0} dx e^{(J-2)x} = \frac{c}{J-2} \tag{15.93}$$

as $x_0 = \frac{(\lambda_0 - \lambda)}{\lambda_0\lambda} < 0$ and, finally,

$$\begin{aligned}
& \left(\mathcal{Q}_2^{(J-2)}(\lambda, \nu) + \underline{\mathcal{Q}}_2^{(J-2)}(\lambda, \nu) \right)^{\frac{1}{2}} \leq \left[C_0^{(1)} + (c'\varepsilon) C^{*(1)} \right] \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right) \\
& < \left(C^{*(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right), \tag{15.94}
\end{aligned}$$

choosing ε sufficiently small, which implies that it can be extended to any ν up to ν_* and to the largest value of $\lambda = \lambda_1$. Therefore the result holds everywhere and the control of the more delicate term of the error is done.

Let us consider now the second term in curly brackets. First we estimate

$$\begin{aligned}
& \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^l \tilde{q}) \cdot (\tilde{W}^{(J-1-l)})|^2 \right)^{\frac{1}{2}} \\
& \leq \int_{\nu_0}^{\nu} d\nu' \tau_+^4 \int_{S(\lambda', \nu')} d\nu |\mathcal{L}_O^l \tilde{q}|^2 |\tilde{W}^{(J-1-l)}|^2 \\
& \leq \int_{\nu_0}^{\nu} d\nu' \tau_+^4 |\mathcal{L}_O^l \tilde{q}|_{4, S(\lambda, \nu')}^2 |\tilde{W}^{(J-1-l)}|_{4, S(\lambda', \nu')}^2 \\
& \leq c_1 \frac{(l+1)!}{(l+1)^\alpha} \frac{e^{(l-1)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^{l+1}} \left(C_1 \frac{(J-l)!}{(J-l)^\alpha} \frac{e^{(J-l-2)(\delta+\Gamma(\lambda', \nu))}}{\rho_{0,1}^{J-l}} \right) \frac{1}{|\lambda'|^3} \cdot \\
& \quad \int_{\nu_0}^{\nu} d\nu' \frac{\tau_+^4}{r^{3+3}} \\
& \leq \left(C_1 \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda'))}}{\rho_{0,1}^J} \right) \left(c_1 \frac{e^{-\delta}}{\rho_{0,1}} \right) \frac{1}{|\lambda'|^3} \left[\frac{J^\alpha}{J!} \frac{(l+1)!}{(l+1)^\alpha} \frac{(J-l)!}{(J-l)^\alpha} \right] \\
& \quad \int_{\nu_0}^{\nu} \frac{d\nu'}{\nu'^2} \\
& \leq \left(C_1 \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right) \left(c_1 c_2 \frac{e^{-\delta}}{\rho_{0,1}} \right) \frac{1}{|\lambda'|^3} \left[\frac{1}{J-1} \frac{J^\alpha}{J!} \frac{(l+1)!}{(l+1)^\alpha} \frac{(J-l)!}{(J-l)^\alpha} \right] \cdot
\end{aligned}$$

Therefore

$$\begin{aligned}
& \left[\sum_{k=0}^{J-1} \sum_{l=0}^{k \wedge (J-2)} \binom{k}{l} \int_{\lambda_0}^{\lambda} d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^l \tilde{q}) \cdot (\tilde{W}^{(J-1-l)})|^2 \right)^{\frac{1}{2}} \right] \\
& \leq \left(C_1 \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda, \nu))}}{\rho_{0,1}^J} \right) \left(c_1 c_2 \frac{e^{-\delta}}{\rho_{0,1}} \right) \cdot \\
& \quad \left[\sum_{k=0}^{J-1} \sum_{l=0}^{k \wedge (J-2)} \binom{k}{l} \left(\frac{1}{J-1} \frac{J^\alpha}{J!} \frac{(l+1)!}{(l+1)^\alpha} \frac{(J-l)!}{(J-l)^\alpha} \right) \int_{\lambda_0}^{\lambda} \frac{d\lambda'}{|\lambda'|^3} e^{-(J-2)(\Gamma(\lambda)-\Gamma(\lambda'))} \right] \\
& \leq \left(C_1 \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda, \nu))}}{\rho_{0,1}^J} \right) \left(c_1 c_2 c_3 \frac{e^{-\delta}}{\rho_{0,1}} \right) \left[\frac{1}{(J-1)} \sum_{k=0}^{J-1} \sum_{l=0}^{k \wedge (J-2)} \frac{k!}{l!(k-l)!} \frac{J^\alpha}{J!} \frac{(l+1)!}{(l+1)^\alpha} \frac{(J-l)!}{(J-l)^\alpha} \right] \\
& \hspace{25em} (15.95)
\end{aligned}$$

Let us estimate the term:

$$\sum_{k=0}^{J-1} \sum_{l=0}^{k \wedge (J-2)} \frac{k!}{J!} \frac{(J-l)!}{(k-l)!} \frac{J^\alpha}{(J-l)^\alpha} \frac{1}{(l+1)^{\alpha-1}} \leq \sum_{k=0}^{J-1} \sum_{l=0}^{k \wedge (J-2)} \frac{J^\alpha}{(J-l)^\alpha} \frac{1}{(l+1)^{\alpha-1}}$$

We estimate this term with $\alpha = 3$,

$$\begin{aligned}
& \sum_{k=0}^{J-1} \sum_{l=0}^{k \wedge (J-2)} \frac{J^\alpha}{(J-l)^\alpha} \frac{1}{(l+1)^{\alpha-1}} \leq C \int_0^{J-1} dk \int_0^k \frac{J^3}{(J-l)^3} \frac{dl}{(l+1)^2} = \\
& C \int_0^{J-1} dk \frac{J^3 \left(\frac{(J+1)^2}{(J-l)^2} + \frac{4(J+1)}{J-l} - \frac{2(J+1)}{l+1} - 6 \ln(J-l) + 6 \ln(l+1) \right)}{2(J+1)^4} \Big|_0^k \leq \\
& C' \int_0^{J-1} dk \frac{J^3 \frac{(J+1)^2}{(J-k)^2}}{2(J+1)^4} \leq C'' \int_0^{J-1} \frac{J+1}{(J-k)^2} dk \leq C'''(J+1)
\end{aligned}$$

And exploiting equation 15.95, we obtain

$$\begin{aligned}
& \sum_{k=0}^{k \wedge (J-2)} \left[\sum_{l=1}^k \binom{k}{l} \int_{\lambda_0}^\lambda d\lambda' \left(\int_{C(\lambda'; [\nu_0, \nu])} \tau_+^4 |(\mathcal{L}_O^l \tilde{q}) \hat{\mathcal{L}}_O^l (\tilde{W}^{(J-1-l)})|^2 \right)^{\frac{1}{2}} \right] \Bigg\} \\
& \leq \left(C^{*(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right) \tag{15.96}
\end{aligned}$$

It is easy to prove that all the other terms of the error can be treated in the same way, hence we can consider proved Theorem 10.1

15.5 Step 5, the estimate of the norms of $\nabla^J \Psi$ in the interior region.

We state first the following lemmas,

Lemma 10.3 *Assuming the following estimate for the norms of $\Psi(\hat{\mathcal{L}}_O^{J-1} W)$ which follow immediately from the \mathcal{Q} norms estimates in Section 15.4,*

$$|\Psi(\hat{\mathcal{L}}_O^{J-1} W)|_{p,S} \leq \left(C_0 \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right), \tag{15.97}$$

the following estimate holds for the norms of $\mathcal{L}_O^{J-1} \Psi(W)$,

$$|\mathcal{L}_O^{J-1} \Psi|_{p,S} \leq \left(C_1 \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right)$$

Lemma 15.1 *from the result of the previous lemma the following estimates hold,*

$$|\nabla_O^{J-1} \Psi|_{p,S} \leq \left(C_2 \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho_{0,1}^J} \right). \tag{15.98}$$

Proof: Their proof is an adapted repetition of the Proof of Lemma 4.2 , see subsection 15.3.

Once we have proved these lemmas we can prove Lemma 10.4 we recall here,

Lemma 10.4

Assuming the norm estimates for the null Riemann component of lemma 15.1 and the estimate of corollary 9.12 for the auxiliary coefficient, h , the following estimates hold:

Theorem 15.1

$$||\lambda|^{\phi(\Psi)} r^{\phi(\Psi)+(J-1)-\frac{2}{p}} \nabla^{J-1} \Psi|_{p,S} \leq \left(C_4 \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta+\Gamma(\lambda))}}{\rho^J} \right) \quad (15.99)$$

with $\rho < \rho_{0,1}$, C_4 of order $O(1)$ for $J \geq 5$, and $O(\varepsilon)$ with $\epsilon_0^2 < \varepsilon < \epsilon_0$ for $J < 5$,¹²⁷

$$\begin{aligned} \phi(\alpha) &= \phi(\beta) = \frac{7}{2}, \quad \phi(\rho) = \phi(\sigma) = 3, \quad \phi(\underline{\beta}) = 2, \quad \phi(\underline{\alpha}) = 1 \\ \underline{\phi}(\alpha) &= \underline{\phi}(\beta) = 0, \quad \underline{\phi}(\rho) = \underline{\phi}(\sigma) = \frac{1}{2}, \quad \underline{\phi}(\underline{\beta}) = \frac{3}{2}, \quad \underline{\phi}(\underline{\alpha}) = \frac{5}{2}. \end{aligned} \quad (15.100)$$

Proof:

Recall that

$$\begin{aligned} |\nabla^J \Psi|^2 &= \sum_{A_1} \sum_{A_2} \dots \sum_{A_J} e_{A_1}^{\mu_1} e_{A_1}^{\nu_1} e_{A_2}^{\mu_2} e_{A_2}^{\nu_2} \dots e_{A_J}^{\mu_J} e_{A_J}^{\nu_J} \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_J} \Psi_\mu \nabla_{\nu_1} \nabla_{\nu_2} \dots \nabla_{\nu_J} \Psi_\nu g^{\mu\nu} \\ &= g^{\mu\nu} \sum_{A_1} \sum_{A_2} \dots \sum_{A_J} \left(e_{A_1}^{\mu_1} e_{A_2}^{\mu_2} \dots e_{A_J}^{\mu_J} \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_J} \Psi_\mu \right) \left(e_{A_1}^{\nu_1} e_{A_2}^{\nu_2} \dots e_{A_J}^{\nu_J} \nabla_{\nu_1} \nabla_{\nu_2} \dots \nabla_{\nu_J} \Psi_\nu \right) \end{aligned} \quad (15.101)$$

and

$$\begin{aligned} |\nabla^J \Psi|^4 &= g^{\mu\nu} \sum_{A_1} \sum_{A_2} \dots \sum_{A_J} \left(e_{A_1}^{\mu_1} e_{A_2}^{\mu_2} \dots e_{A_J}^{\mu_J} \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_J} \Psi_\mu \right) \left(e_{A_1}^{\nu_1} e_{A_2}^{\nu_2} \dots e_{A_J}^{\nu_J} \nabla_{\nu_1} \nabla_{\nu_2} \dots \nabla_{\nu_J} \Psi_\nu \right) \\ &\quad g^{\tau\sigma} \sum_{B_1} \sum_{B_2} \dots \sum_{B_J} \left(e_{B_1}^{\mu_1} e_{B_2}^{\mu_2} \dots e_{B_J}^{\mu_J} \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_J} \Psi_\tau \right) \left(e_{B_1}^{\nu_1} e_{B_2}^{\nu_2} \dots e_{B_J}^{\nu_J} \nabla_{\nu_1} \nabla_{\nu_2} \dots \nabla_{\nu_J} \Psi_\sigma \right) \\ &= g^{\mu\nu} g^{\tau\sigma} \sum_{A_1} \sum_{A_2} \dots \sum_{A_J} \sum_{B_1} \sum_{B_2} \dots \sum_{B_J} \left(e_{A_1}^{\mu_1} e_{A_2}^{\mu_2} \dots e_{A_J}^{\mu_J} \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_J} \Psi_\mu \right) \left(e_{B_1}^{\mu_1} e_{B_2}^{\mu_2} \dots e_{B_J}^{\mu_J} \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_J} \Psi_\tau \right) \\ &\quad \cdot \left[\left(e_{A_1}^{\nu_1} e_{A_2}^{\nu_2} \dots e_{A_J}^{\nu_J} \nabla_{\nu_1} \nabla_{\nu_2} \dots \nabla_{\nu_J} \Psi_\nu \right) \left(e_{B_1}^{\mu_1} e_{B_2}^{\mu_2} \dots e_{B_J}^{\mu_J} \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_J} \Psi_\tau \right) \right] \\ &\leq |g^{\mu\nu} g^{\tau\sigma} \sum_{A_1} \sum_{A_2} \dots \sum_{A_J} \sum_{B_1} \sum_{B_2} \dots \sum_{B_J} \left(e_{A_1}^{\mu_1} e_{A_2}^{\mu_2} \dots e_{A_J}^{\mu_J} \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_J} \Psi_\mu \right) \left(e_{B_1}^{\mu_1} e_{B_2}^{\mu_2} \dots e_{B_J}^{\mu_J} \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_J} \Psi_\tau \right)| \end{aligned}$$

¹²⁷If $J \geq 1$, for $J = 0$ ρ becomes $\rho - \underline{\rho}$, see [Kl-Ni2].

$$\begin{aligned}
& \cdot \sup_{A_1, \dots, A_J, B_1, \dots, B_J} \left| \left[\left(e_{A_1}^{\nu_1} e_{A_2}^{\nu_2} \cdot \cdot e_{A_J}^{\nu_J} \nabla_{\nu_1} \nabla_{\nu_2} \cdot \cdot \nabla_{\nu_J} \Psi_\nu \right) \left(e_{B_1}^{\mu_1} e_{B_2}^{\mu_2} \cdot \cdot e_{B_J}^{\mu_J} \nabla_{\mu_1} \nabla_{\mu_2} \cdot \cdot \nabla_{\mu_J} \Psi_\tau \right) \right] \right| \\
& \leq |g^{\mu\nu} g^{\tau\sigma} \sum_{A_1} \sum_{A_2} \cdot \cdot \sum_{A_J} \sum_{B_1} \sum_{B_2} \cdot \cdot \sum_{B_J} \left(e_{A_1}^{\mu_1} e_{A_2}^{\mu_2} \cdot \cdot e_{A_J}^{\mu_J} \nabla_{\mu_1} \nabla_{\mu_2} \cdot \cdot \nabla_{\mu_J} \Psi_\mu \right) \left(e_{B_1}^{\mu_1} e_{B_2}^{\mu_2} \cdot \cdot e_{B_J}^{\mu_J} \nabla_{\mu_1} \nabla_{\mu_2} \cdot \cdot \nabla_{\mu_J} \Psi_\tau \right)| \\
& \cdot |\nabla^J \Psi|^2. \tag{15.102}
\end{aligned}$$

Therefore

$$\begin{aligned}
|\nabla^J \Psi|^2 & \leq c |g^{\mu\nu} g^{\tau\sigma} \sum_{A_1} \sum_{A_2} \cdot \cdot \sum_{A_J} \sum_{B_1} \sum_{B_2} \cdot \cdot \sum_{B_J} \left(e_{A_1}^{\mu_1} e_{A_2}^{\mu_2} \cdot \cdot e_{A_J}^{\mu_J} \nabla_{\mu_1} \nabla_{\mu_2} \cdot \cdot \nabla_{\mu_J} \Psi_\mu \right) \left(e_{B_1}^{\mu_1} e_{B_2}^{\mu_2} \cdot \cdot e_{B_J}^{\mu_J} \nabla_{\mu_1} \nabla_{\mu_2} \cdot \cdot \nabla_{\mu_J} \Psi_\tau \right)| \\
& \leq c_1^J \frac{1}{r^{2J}} \sup_{\mu\nu\tau\sigma} |g^{\mu\nu} g^{\tau\sigma}| \sup_\mu \left| \left(\sum_{i_1} \sum_{i_2} \cdot \cdot \sum_{i_J} O_{i_1}^{\mu_1} O_{i_2}^{\mu_2} \cdot \cdot O_{i_J}^{\mu_J} \nabla_{\mu_1} \nabla_{\mu_2} \cdot \cdot \nabla_{\mu_J} \Psi_\mu \right) \right| \sup_\nu \left| \left(\sum_{l_1} \sum_{l_2} \cdot \cdot \sum_{l_J} O_{l_1}^{\nu_1} O_{l_2}^{\nu_2} \cdot \cdot O_{l_J}^{\nu_J} \nabla_{\nu_1} \nabla_{\nu_2} \cdot \cdot \nabla_{\nu_J} \Psi_\nu \right) \right|.
\end{aligned}$$

Denoting

$$O^\mu \equiv \sum_l O_l^\mu$$

we need a more explicit expression of

$$\sup_\mu |O^{\mu_1} O^{\mu_2} \cdot \cdot O^{\mu_J} \nabla_{\mu_1} \nabla_{\mu_2} \cdot \cdot \nabla_{\mu_J} \Psi_\mu|.$$

$$\begin{aligned}
& O^{\mu_1} O^{\mu_2} \cdot \cdot O^{\mu_J} \nabla_{\mu_1} \nabla_{\mu_2} \cdot \cdot \nabla_{\mu_J} \Psi_\mu \\
& = O^{\mu_1} O^{\mu_2} \cdot \cdot O^{\mu_J} \delta_{\mu_1}^{\rho_1} \delta_{\mu_2}^{\rho_2} \cdot \cdot \delta_{\mu_J}^{\rho_J} \nabla_{\rho_1} \nabla_{\rho_2} \cdot \cdot \nabla_{\rho_J} \Psi_\mu = O^{\mu_1} O^{\mu_2} \cdot \cdot O^{\mu_J} \delta_{\mu_1}^{\rho_1} \nabla_{\rho_1} \delta_{\mu_2}^{\rho_2} \nabla_{\rho_2} \cdot \cdot \delta_{\mu_J}^{\rho_J} \nabla_{\rho_J} \Psi_\mu \\
& = O^{\mu_1} O^{\mu_2} \cdot \cdot O^{\mu_J} \left[\sum_{A_1} \sum_{A_2} \cdot \cdot \sum_{A_J} (e_{A_1})_{\mu_1} e_{A_1}^{\rho_1} \nabla_{\rho_1} (e_{A_2})_{\mu_2} e_{A_2}^{\rho_2} \nabla_{\rho_2} \cdot \cdot (e_{A_J})_{\mu_J} e_{A_J}^{\rho_J} \nabla_{\rho_J} \Psi_\mu \right] \\
& = O^{\mu_2} \cdot \cdot O^{\mu_J} \left[\sum_{A_1} \sum_{A_2} \cdot \cdot \sum_{A_J} \mathbf{g}(O, e_{A_1}) \nabla_{A_1} (e_{A_2})_{\mu_2} \nabla_{A_2} \cdot \cdot (e_{A_J})_{\mu_J} \nabla_{A_J} \Psi_\mu \right]. \tag{15.103}
\end{aligned}$$

We have,

$$\begin{aligned}
& \sup_\mu |O^{\mu_1} O^{\mu_2} \cdot \cdot O^{\mu_J} \nabla_{\mu_1} \nabla_{\mu_2} \cdot \cdot \nabla_{\mu_J} \Psi_\mu| \\
& \leq |O|^{J-1} \left(\sup_A |\mathbf{g}(O, e_A)| \right) 2^J \sup_\mu \sup_{A_2, A_3, \dots, A_J} |\nabla_{A_1} (e_{A_2})_{\mu_2} \nabla_{A_2} \cdot \cdot (e_{A_J})_{\mu_J} \nabla_{A_J} \Psi_\mu| \\
& \leq C_1^J \sup_\mu \sup_{A_2, A_3, \dots, A_J; \mu_2, \mu_3, \dots, \mu_J} |\nabla_{A_1} (e_{A_2})_{\mu_2} \nabla_{A_2} \cdot \cdot (e_{A_J})_{\mu_J} \nabla_{A_J} \Psi_\mu|. \tag{15.104}
\end{aligned}$$

We are left with estimating

$$|\nabla_{A_1} (e_{A_2})_{\mu_2} \nabla_{A_2} \cdot \cdot (e_{A_J})_{\mu_J} \nabla_{A_J} \Psi_\mu|, \tag{15.105}$$

This goes basically as Lemma 4.1, the main difference being that now the derivatives are $\nabla_A \mathcal{O}$ which are estimated in term of the $\nabla_O \mathcal{O}$ and, therefore, with a different ρ namely a $\rho_{0,3}$ such that $\rho_{0,3} > \rho$

This would imply, finally, that the C_1^J in front is controlled by the factor $\left(\frac{\rho}{\rho_{0,3}}\right)^J$.

Let us reproduce the Lemma 4.1 applied to this case.

We write $\nabla_{A_1}(e_{A_2})_{\mu_2} \nabla_{A_2} \cdots (e_{A_J})_{\mu_J} \nabla_{A_J} \Psi$, omitting the μ indices, as

$$\nabla_{A_1}(e_{A_2}) \nabla_{A_2} \cdots (e_{A_J}) \nabla_{A_J} \Psi .$$

As the e_A are endowed with a given name, therefore numbered, we think at each one as defining a “slot” namely the position at its left, therefore there are $J-1$ slots associated to the $(J-1)$ e_A ’s we are considering; we add an extra slot which corresponds to the position immediately at the left of Ψ . Therefore we have J slots and we imagine to distribute in all the possible ways the J ∇_{A_i} in these slots. Clearly in this way we are counting more than the possible terms we can really obtain as each ∇_A cannot operate on the e_A ’s at its left. Assuming the described distribution of the ∇ ’s we have (the equality is formal and it has to be interpreted as an upper bounds when we consider the norms),

$$\begin{aligned} & \nabla_{A_1}(e_{A_2}) \nabla_{A_2} \cdots (e_{A_J}) \nabla_{A_J} \Psi \\ \text{“} = \text{”} & \left\{ \sum_{\gamma_1, \gamma_2, \dots, \gamma_{J-2}, \gamma_J}^{\sum_{s=1}^J \gamma_s = J; \gamma_J \geq 1} \frac{J!}{\gamma_1! \gamma_2! \cdots \gamma_{J-1}! \gamma_J!} (\nabla_A^{\gamma_1} e_{A_2}) (\nabla_A^{\gamma_2} e_{A_3}) \cdots (\nabla_A^{\gamma_{J-1}} e_{A_J}) (\nabla_A^{\gamma_J} \Psi) \right\} . \end{aligned}$$

We denote $\gamma_{J-1} = q$ and rewrite the previous expression as

$$\begin{aligned} & \nabla_{A_1}(e_{A_2}) \nabla_{A_2} \cdots (e_{A_J}) \nabla_{A_J} \Psi \\ \text{“} = \text{”} & \left\{ \sum_{q=1}^J \binom{J}{q} \nabla_A^q \Psi \sum_{\gamma_1, \gamma_2, \dots, \gamma_{J-2}}^{\sum_{s=1}^{J-1} \gamma_s = (J-q)} \frac{(J-q)!}{\gamma_1! \gamma_2! \cdots \gamma_{J-1}!} (\nabla_A^{\gamma_1} e_{A_2}) (\nabla_A^{\gamma_2} e_{A_3}) \cdots (\nabla_A^{\gamma_{J-1}} e_{A_J}) \right\} . \end{aligned}$$

Denoting $k = J - q$ we rewrite the previous expression as

$$\begin{aligned} & \nabla_{A_1}(e_{A_2}) \nabla_{A_2} \cdots (e_{A_J}) \nabla_{A_J} \Psi \\ \text{“} = \text{”} & \left\{ \sum_{k=0}^J \binom{J}{k} \nabla_A^{J-k} \Psi \sum_{\gamma_1, \gamma_2, \dots, \gamma_{J-1}}^{\sum_{s=1}^{J-1} \gamma_s = k} \frac{k!}{\gamma_1! \gamma_2! \cdots \gamma_{J-1}!} (\nabla_A^{\gamma_1} e_{A_2}) (\nabla_A^{\gamma_2} e_{A_3}) \cdots (\nabla_A^{\gamma_{J-1}} e_{A_J}) \right\} \\ & = \nabla_A^J \Psi + \left\{ \sum_{k=1}^J \binom{J}{k} \nabla_A^{J-k} \Psi \sum_{\gamma_1, \gamma_2, \dots, \gamma_{J-1}}^{\sum_{s=1}^{J-1} \gamma_s = k} \frac{k!}{\gamma_1! \gamma_2! \cdots \gamma_{J-1}!} (\nabla_A^{\gamma_1} e_{A_2}) (\nabla_A^{\gamma_2} e_{A_3}) \cdots (\nabla_A^{\gamma_{J-1}} e_{A_J}) \right\} . \end{aligned}$$

Going to the norms, the estimate becomes

$$\begin{aligned} & |\nabla_{A_1}(e_{A_2}) \nabla_{A_2} \cdots (e_{A_J}) \nabla_{A_J} \Psi|_{p,S} \leq |\nabla_A^J \Psi|_{p,S} \\ & + \left\{ \sum_{k=1}^J \binom{J}{k} |\nabla_A^{J-k} \Psi|_{p,S} \sum_{\gamma_1, \gamma_2, \dots, \gamma_{J-1}}^{\sum_{s=1}^{J-1} \gamma_s = k} \frac{k!}{\gamma_1! \gamma_2! \cdots \gamma_{J-1}!} \left[|(\nabla_A^{\gamma_1} e_{A_2})|_{\infty,S} |(\nabla_A^{\gamma_2} e_{A_3})|_{\infty,S} \cdots |(\nabla_A^{\gamma_{J-1}} e_{A_J})|_{\infty,S} \right] \right\} . \end{aligned} \tag{15.106}$$

We use now the results of theorem 9.13 and of corollary 9.14 for the norms with ∇_A^J of the connection coefficients, and of the the Riemann null components estimates and the norms ∇_A^J of the e_A versors,^{128 129}

$$\begin{aligned} |\nabla_A^J \Psi|_{p,S} &\leq C \frac{(J+1)!}{(J+1)^\alpha} \frac{e^{(J-1)\delta} e^{(J-1)\Gamma(\lambda)}}{\rho_{0,3}^{J+1}} \\ |r^{J+\phi(\mathcal{O})} \nabla_A^J e_{A_i}|_{\infty,S} &\leq C \frac{(J+1)!}{(J+1)^\alpha} \frac{e^{(J-1)\delta} e^{(J-1)\Gamma(\lambda)}}{\rho_{0,3}^{J+1}}. \end{aligned} \quad (15.107)$$

From these estimates it follows that

$$\begin{aligned} \delta\left(\sum_{s=1}^{J-1} \gamma_s = k\right) &\left[|(\nabla_A^{\gamma_1} e_{A_2})|_{\infty,S} |(\nabla_A^{\gamma_2} e_{A_3})|_{\infty,S} \cdots |(\nabla_A^{\gamma_{J-1}} e_{A_J})|_{\infty,S} \right] \\ &\leq \frac{e^{k(\delta+\Gamma)}}{\rho_{0,3}^{k+J-1}} \frac{(\gamma_1+1)! (\gamma_2+1)! \cdots (\gamma_{J-1}+1)! e^{-(\delta+\Gamma)(\epsilon(\gamma_1)+\epsilon(\gamma_2)+\cdots+\epsilon(\gamma_{J-1}))}}{(1+\gamma_1)^\alpha (1+\gamma_2)^\alpha \cdots (1+\gamma_{J-1})^\alpha} \end{aligned}$$

and we can rewrite the previous estimate, as

$$\begin{aligned} |\nabla_{A_1}(e_{A_2}) \nabla_{A_2} \cdots (e_{A_J}) \nabla_{A_J} \Psi|_{p,S} &\leq |\nabla_A^J \Psi|_{p,S} + \left\{ \sum_{k=1}^{J-2} \binom{J-1}{k} |\nabla_A^{J-k} \Psi|_{p,S} \frac{e^{k(\delta+\Gamma)}}{\rho_{0,3}^k} k! \cdot \right. \\ &\cdot \left[\sum_{\gamma_1, \gamma_2, \dots, \gamma_{J-2}}^{\sum_{s=1}^{J-2} \gamma_s = k} \frac{C^{(\epsilon(\gamma_1)+\epsilon(\gamma_2)+\cdots+\epsilon(\gamma_{J-1}))} e^{-(\delta_0+\Gamma_0(\lambda))(\epsilon(\gamma_1)+\epsilon(\gamma_2)+\cdots+\epsilon(\gamma_{J-2}))}}{(1+\gamma_1)^\alpha (1+\gamma_2)^\alpha \cdots (1+\gamma_{J-1})^\alpha} \right] \right\}. \end{aligned}$$

Applying the previous estimates for the various terms we have

$$|\nabla_{A_1}(e_{A_2}) \nabla_{A_2} \cdots (e_{A_J}) \nabla_{A_J} \Psi|_{p,S} \leq \left\{ C \frac{(J+1)!}{(J+1)^\alpha} \frac{e^{(J-1)(\delta+\Gamma(\lambda))}}{\rho_{0,3}^{(J+1)}} \right. \quad (15.108)$$

$$\left. + \sum_{k=1}^{J-2} \binom{J-1}{k} C \frac{(J+1-k)!}{(J+1-k)^\alpha} \frac{e^{((J-k)-1)(\delta+\Gamma)}}{\rho_{0,3}^{J+1-k}} \frac{e^{k(\delta+\Gamma(\lambda))}}{\rho_{0,3}^k} k! |\mathcal{F}| \right\} \quad (15.109)$$

where

$$[\mathcal{F}] \equiv \left[\sum_{\gamma_1, \gamma_2, \dots, \gamma_{J-1}}^{\sum_{s=1}^{J-1} \gamma_s = k} \frac{C^{(\epsilon(\gamma_1)+\epsilon(\gamma_2)+\cdots+\epsilon(\gamma_{J-2}))} e^{-(\delta+\Gamma(\lambda))(\epsilon(\gamma_1)+\epsilon(\gamma_2)+\cdots+\epsilon(\gamma_{J-2}))}}{\rho_{0,3}^{J-1} (1+\gamma_1)^\alpha (1+\gamma_2)^\alpha \cdots (1+\gamma_{J-1})^\alpha} \right] \quad (15.110)$$

¹²⁸In fact the weight factors in the following estimates are for some components a little better, see Theorem ?? of Section 7, but this is irrelevant here.

¹²⁹The estimate in the third line follows as (assuming $r = 1$) although the O generators are at the metric level, nevertheless in the estimates of the angular derivatives of γ_{ab} and of X^a there is a loss of derivatives, see subsection 5.3, which implies the same estimates as for the connection coefficients.

Therefore we rewrite 15.109 as

$$\begin{aligned}
|\nabla_{A_1}(e_{A_2})\nabla_{A_2}\cdots(e_{A_J})\nabla_{A_J}\Psi|_{p,S} &\leq \left(C_3\frac{(J+1)!}{(J+1)^\alpha}\frac{e^{(J-1)(\delta+\Gamma(\lambda))}}{\rho_{0,3}^{(J+1)}}\right)\left\{1+\sum_{k=1}^{J-1}\binom{J-1}{k}\frac{(J+1-k)!k!}{(J+1)!}\frac{(J+1)^\alpha}{(J-k)^\alpha}|\mathcal{F}|\right\} \\
&\leq \left(C\frac{(J+1)!}{(J+1)^\alpha}\frac{e^{(J-1)(\delta+\Gamma(\lambda))}}{\rho_{0,3}^{(J+1)}}\right)\left\{1+\sum_{k=1}^{J-1}\frac{(J+1-k)(J-k)}{(J+1)J}\frac{(J+1)^\alpha}{(J-k)^\alpha}|\mathcal{F}|\right\} \\
&\leq \left(C\frac{(J+1)!}{(J+1)^\alpha}\frac{e^{(J-1)(\delta+\Gamma(\lambda))}}{\rho_{0,3}^{(J+1)}}\right)\left\{1+\sum_{k=1}^{J-1}\frac{(J+1)^{\alpha-2}}{(J-k)^{\alpha-2}}|\mathcal{F}|\right\}. \tag{15.111}
\end{aligned}$$

Provide α is sufficiently large we have

$$|\mathcal{F}| \leq C^{J-1} \tag{15.112}$$

so that

$$\begin{aligned}
|\nabla_{A_1}(e_{A_2})\nabla_{A_2}\cdots(e_{A_J})\nabla_{A_J}\Psi|_{p,S} &\leq C\left(\frac{(J+1)!}{(J+1)^\alpha}\frac{e^{(J-1)(\delta+\Gamma(\lambda))}}{\rho_{0,4}^{(J+1)}}\right)\left(\frac{\rho_{0,4}}{\rho_{0,3}}\right)^{J+1}C^{J+1} \\
&\leq C\left(\left(\frac{\rho_{0,4}}{\rho_{0,3}}\right)^{J+1}C^{J+1}\right)\left(\frac{(J+1)!}{(J+1)^\alpha}\frac{e^{(J-1)(\delta+\Gamma(\lambda))}}{\rho_{0,4}^{J+1}}\right) \leq C\left(\frac{(J+1)!}{(J+1)^\alpha}\frac{e^{(J-1)(\delta+\Gamma(\lambda))}}{\rho_{0,4}^{J+1}}\right). \tag{15.113}
\end{aligned}$$

Provided $\rho_{0,4}$ sufficiently small. Therefore from 15.104 we have

$$\begin{aligned}
&\sup_{\mu} |O^{\mu_1}O^{\mu_2}\cdots O^{\mu_J}\nabla_{\mu_1}\nabla_{\mu_2}\cdots\nabla_{\mu_J}\Psi_{\mu}| \\
&\leq |O|^{J-1}\left(\sup_A |\mathbf{g}(O, e_A)|\right)2^J\sup_{\mu}\sup_{A_2, A_3, \dots, A_J} |\nabla_{A_1}(e_{A_2})_{\mu_2}\nabla_{A_2}\cdots(e_{A_J})_{\mu_J}\nabla_{A_J}\Psi_{\mu}| \\
&\leq C^J\sup_{\mu}\sup_{A_2, A_3, \dots, A_J; \mu_2, \mu_3, \dots, \mu_J} |\nabla_{A_1}(e_{A_2})_{\mu_2}\nabla_{A_2}\cdots(e_{A_J})_{\mu_J}\nabla_{A_J}\Psi_{\mu}| \\
&\leq C^JC\left(\frac{(J+1)!}{(J+1)^\alpha}\frac{e^{(J-1)(\delta+\Gamma(\lambda))}}{\rho_{0,4}^{J+1}}\right) \leq C\left(\frac{(J+1)!}{(J+1)^\alpha}\frac{e^{(J-1)(\delta+\Gamma(\lambda))}}{\rho_{0,5}^{J+1}}\right) \tag{15.114}
\end{aligned}$$

for suitable $\rho_{0,5} < \rho_{0,4}$ and

$$\begin{aligned}
|\nabla^J\Psi|^2 &\leq c|g^{\mu\nu}g^{\tau\sigma}\sum_{A_1}\sum_{A_2}\cdots\sum_{A_J}\sum_{B_1}\sum_{B_2}\cdots\sum_{B_J}\left(e_{A_1}^{\mu_1}e_{A_2}^{\mu_2}\cdots e_{A_J}^{\mu_J}\nabla_{\mu_1}\nabla_{\mu_2}\cdots\nabla_{\mu_J}\Psi_{\mu}\right)\left(e_{B_1}^{\mu_1}e_{B_2}^{\mu_2}\cdots e_{B_J}^{\mu_J}\nabla_{\mu_1}\nabla_{\mu_2}\cdots\nabla_{\mu_J}\Psi_{\tau}\right)| \\
&\leq c_1^J\frac{1}{r^{2J}}\sup_{\mu\nu\tau\sigma}|g^{\mu\nu}g^{\tau\sigma}|\sup_{\mu}\left|\left(\sum_{i_1}\sum_{i_2}\cdots\sum_{i_J}O_{i_1}^{\mu_1}O_{i_2}^{\mu_2}\cdots O_{i_J}^{\mu_J}\nabla_{\mu_1}\nabla_{\mu_2}\cdots\nabla_{\mu_J}\Psi_{\mu}\right)\right|\sup_{\nu}\left|\left(\sum_{l_1}\sum_{l_2}\cdots\sum_{l_J}O_{l_1}^{\nu_1}O_{l_2}^{\nu_2}\cdots O_{l_J}^{\nu_J}\nabla_{\nu_1}\nabla_{\nu_2}\cdots\nabla_{\nu_J}\Psi_{\nu}\right)\right| \\
&\leq C\left(\frac{(J+1)!}{(J+1)^\alpha}\frac{e^{(J-1)(\delta+\Gamma(\lambda))}}{\rho^{J+1}}\right) \tag{15.115}
\end{aligned}$$

with $\rho < \rho_{0,1}$, where we omit the weight factors $|\lambda|^{\alpha}r^{\beta}$ as they come out in a obvious way. Therefore Lemma 10.4 has been proven.

To summarize, the previous lemma proves that, apart some technicalities, we have an estimate of the following kind

$$|r^J \nabla^J \Psi| \leq C^J |\hat{\mathcal{L}}_O^J \Psi|. \quad (15.116)$$

To control this C^J factor we have to lower the ρ of the estimate of $|\hat{\mathcal{L}}_O^J \Psi|$.

15.6 Proof of Lemma 4.3

We repeat here the statement of Lemma 4.3:

Lemma 4.3 *Let us assume the estimates of theorem 4.2, then the null Riemann components Ψ satisfy the following estimates for $p + q = J$ on \mathcal{K} with $\rho_1 < \rho < \rho_{0,1}$ depending only the first s derivatives of the initial data, with $s \leq 7$ and with a suitable choice of the $\tilde{C}^{(1)}$ constant of order $O(\varepsilon)$ for $J \leq 7^{130}$ and $O(1)$ otherwise.,¹³¹*

$$||\lambda|^{\phi(\Psi)} r^{\phi(\Psi) + k - \frac{2}{p}} D_S^p \nabla^k \Psi|_{L^p(S)} \leq \tilde{C}^{(1)} e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^\alpha} \frac{1}{\rho_1^J}, \quad (15.117)$$

$$\begin{aligned} \phi(\alpha) = \phi(\beta) = \frac{7}{2}, \quad \phi(\rho) = \phi(\sigma) = 3, \quad \phi(\underline{\beta}) = 2, \quad \phi(\underline{\alpha}) = 1 \\ \underline{\phi}(\alpha) = \underline{\phi}(\beta) = 0, \quad \underline{\phi}(\rho) = \underline{\phi}(\sigma) = \frac{1}{2}, \quad \underline{\phi}(\underline{\beta}) = \frac{3}{2}, \quad \underline{\phi}(\underline{\alpha}) = \frac{5}{2}. \end{aligned} \quad (15.118)$$

Proof: Preliminary we recall Theorem 4.2, whose Proofs are again an adapted repetition of the Proof of Lemma 4.2.

Theorem 4.2 *Under the assumptions 3.22 and 3.23 on the mixed derivatives of the initial data connection coefficients on $C_0 \cup \underline{C}_0$, the following estimates hold, with $\tilde{C}^{(1)}$ of order $O(\varepsilon)$ for any $J \leq 7$ and order $O(1)$ otherwise.*

$$||\lambda|^{\phi(\Psi)} r^{\phi(\Psi) - \frac{2}{p}} \Psi(\hat{\mathcal{L}}_S^p \hat{\mathcal{L}}_O^k W)|_{p,S(\lambda, \nu)} \leq \tilde{C}^{(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta + \Gamma(\lambda))}}{\rho_{0,1}^J}. \quad (15.119)$$

$$||\lambda|^{\phi(\Psi)} r^{\phi(\Psi) - \frac{2}{p}} \mathcal{L}_S^p \mathcal{L}_O^k \Psi|_{p,S} \leq \tilde{C}^{(1)} \frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta + \Gamma(\lambda))}}{\rho_{0,1}^J}. \quad (15.120)$$

Let us now prove lemma 4.3, for the sake of simplicity we omit the $r^{\phi(\cdot)}$ term. From the result of the previous lemma the following estimates hold,

¹³⁰If $J \geq 1$, for $J = 0$ ρ becomes $\rho - \underline{\rho}$, see [Kl-Ni2].

¹³¹Here ρ_1 plays the same role of ρ for the angular derivatives estimates.

The proof is a repetition of lemma 10.4, recall, in fact that

$$\begin{aligned}
|D_S^p \nabla^k \Psi|^2 &= \sum_{A_1} \sum_{A_2} \dots \sum_{A_k} e_{A_1}^{\mu_1} e_{A_1}^{\nu_1} e_{A_2}^{\mu_2} e_{A_2}^{\nu_2} \dots e_{A_k}^{\mu_k} e_{A_k}^{\nu_k} D_S^p \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_k} \Psi_\mu D_S^p \nabla_{\nu_1} \nabla_{\nu_2} \dots \nabla_{\nu_k} \Psi_\nu g^{\mu\nu} \\
&= g^{\mu\nu} \sum_{A_1} \sum_{A_2} \dots \sum_{A_k} \left(e_{A_1}^{\mu_1} e_{A_2}^{\mu_2} \dots e_{A_k}^{\mu_k} D_S^p \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_k} \Psi_\mu \right) \left(e_{A_1}^{\nu_1} e_{A_2}^{\nu_2} \dots e_{A_k}^{\nu_k} D_S^p \nabla_{\nu_1} \nabla_{\nu_2} \dots \nabla_{\nu_k} \Psi_\nu \right)
\end{aligned} \tag{15.121}$$

and

$$\begin{aligned}
|D_S^p \nabla^J \Psi|^4 &= g^{\mu\nu} \sum_{A_1} \sum_{A_2} \dots \sum_{A_k} \left(e_{A_1}^{\mu_1} e_{A_2}^{\mu_2} \dots e_{A_k}^{\mu_k} D_S^p \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_k} \Psi_\mu \right) \left(e_{A_1}^{\nu_1} e_{A_2}^{\nu_2} \dots e_{A_k}^{\nu_k} D_S^p \nabla_{\nu_1} \nabla_{\nu_2} \dots \nabla_{\nu_k} \Psi_\nu \right) \\
&\quad g^{\tau\sigma} \sum_{B_1} \sum_{B_2} \dots \sum_{B_k} \left(e_{B_1}^{\mu_1} e_{B_2}^{\mu_2} \dots e_{B_k}^{\mu_k} D_S^p \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_k} \Psi_\tau \right) \left(e_{B_1}^{\nu_1} e_{B_2}^{\nu_2} \dots e_{B_k}^{\nu_k} D_S^p \nabla_{\nu_1} \nabla_{\nu_2} \dots \nabla_{\nu_k} \Psi_\sigma \right) \\
&= g^{\mu\nu} g^{\tau\sigma} \sum_{A_1} \sum_{A_2} \dots \sum_{A_k} \sum_{B_1} \sum_{B_2} \dots \sum_{B_k} \left(e_{A_1}^{\mu_1} e_{A_2}^{\mu_2} \dots e_{A_k}^{\mu_k} D_S^p \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_k} \Psi_\mu \right) \left(e_{B_1}^{\mu_1} e_{B_2}^{\mu_2} \dots e_{B_k}^{\mu_k} D_S^p \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_k} \Psi_\tau \right) \\
&\quad \cdot \left[\left(e_{A_1}^{\nu_1} e_{A_2}^{\nu_2} \dots e_{A_k}^{\nu_k} D_S^p \nabla_{\nu_1} \nabla_{\nu_2} \dots \nabla_{\nu_k} \Psi_\nu \right) \left(e_{B_1}^{\mu_1} e_{B_2}^{\mu_2} \dots e_{B_k}^{\mu_k} D_S^p \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_k} \Psi_\tau \right) \right] \\
&\leq |g^{\mu\nu} g^{\tau\sigma} \sum_{A_1} \sum_{A_2} \dots \sum_{A_k} \sum_{B_1} \sum_{B_2} \dots \sum_{B_k} \left(e_{A_1}^{\mu_1} e_{A_2}^{\mu_2} \dots e_{A_k}^{\mu_k} D_S^p \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_k} \Psi_\mu \right) \left(e_{B_1}^{\mu_1} e_{B_2}^{\mu_2} \dots e_{B_k}^{\mu_k} D_S^p \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_k} \Psi_\tau \right)| \\
&\quad \cdot \sup_{A_1, \dots, A_k, B_1, \dots, B_k} \left| \left[\left(e_{A_1}^{\nu_1} e_{A_2}^{\nu_2} \dots e_{A_k}^{\nu_k} D_S^p \nabla_{\nu_1} \nabla_{\nu_2} \dots \nabla_{\nu_k} \Psi_\nu \right) \left(e_{B_1}^{\mu_1} e_{B_2}^{\mu_2} \dots e_{B_k}^{\mu_k} D_S^p \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_k} \Psi_\tau \right) \right] \right| \\
&\leq |g^{\mu\nu} g^{\tau\sigma} \sum_{A_1} \sum_{A_2} \dots \sum_{A_k} \sum_{B_1} \sum_{B_2} \dots \sum_{B_k} \left(e_{A_1}^{\mu_1} e_{A_2}^{\mu_2} \dots e_{A_k}^{\mu_k} D_S^p \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_k} \Psi_\mu \right) \left(e_{B_1}^{\mu_1} e_{B_2}^{\mu_2} \dots e_{B_k}^{\mu_k} D_S^p \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_k} \Psi_\tau \right)| \\
&\quad \cdot |D_S^p \nabla^k \Psi|^2.
\end{aligned} \tag{15.122}$$

Therefore

$$\begin{aligned}
|D_S^p \nabla^k \Psi|^2 &\leq c |g^{\mu\nu} g^{\tau\sigma} \sum_{A_1} \sum_{A_2} \dots \sum_{A_k} \sum_{B_1} \sum_{B_2} \dots \sum_{B_k} \left(e_{A_1}^{\mu_1} e_{A_2}^{\mu_2} \dots e_{A_k}^{\mu_k} D_S^p \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_k} \Psi_\mu \right) \left(e_{B_1}^{\mu_1} e_{B_2}^{\mu_2} \dots e_{B_k}^{\mu_k} D_S^p \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_k} \Psi_\tau \right)| \\
&\leq c_1^k \frac{1}{r^{2k}} \sup_{\mu\nu\tau\sigma} |g^{\mu\nu} g^{\tau\sigma}| \sup_{\mu} \left| \left(\sum_{i_1} \sum_{i_2} \dots \sum_{i_k} O_{i_1}^{\mu_1} O_{i_2}^{\mu_2} \dots O_{i_k}^{\mu_k} D_S^p \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_k} \Psi_\mu \right) \right| \sup_{\nu} \left| \left(\sum_{l_1} \sum_{l_2} \dots \sum_{l_k} O_{l_1}^{\nu_1} O_{l_2}^{\nu_2} \dots O_{l_k}^{\nu_k} D_S^p \nabla_{\nu_1} \nabla_{\nu_2} \dots \nabla_{\nu_k} \Psi_\nu \right) \right|.
\end{aligned}$$

Denoting

$$O^\mu \equiv \sum_l O_l^\mu,$$

we need a more explicit expression of

$$\sup_{\mu} |O^{\mu_1} O^{\mu_2} \dots O^{\mu_k} D_S^p \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_k} \Psi_\mu|.$$

$$\begin{aligned}
&O^{\mu_1} O^{\mu_2} \dots O^{\mu_k} D_S^p \nabla_{\mu_1} \nabla_{\mu_2} \dots \nabla_{\mu_k} \Psi_\mu \\
&= O^{\mu_1} O^{\mu_2} \dots O^{\mu_k} D_S^p \delta_{\mu_1}^{\rho_1} \delta_{\mu_2}^{\rho_2} \dots \delta_{\mu_k}^{\rho_k} \nabla_{\rho_1} \nabla_{\rho_2} \dots \nabla_{\rho_k} \Psi_\mu = O^{\mu_1} O^{\mu_2} \dots O^{\mu_k} D_S^p \delta_{\mu_1}^{\rho_1} \nabla_{\rho_1} \delta_{\mu_2}^{\rho_2} \nabla_{\rho_2} \dots \delta_{\mu_k}^{\rho_k} \nabla_{\rho_k} \Psi_\mu
\end{aligned}$$

$$\begin{aligned}
&= O^{\mu_1} O^{\mu_2} \dots O^{\mu_k} \left[\sum_{A_1} \sum_{A_2} \dots \sum_{A_k} D_S^p(e_{A_1})_{\mu_1} e_{A_1}^{\rho_1} \nabla_{\rho_1}(e_{A_2})_{\mu_2} e_{A_2}^{\rho_2} \nabla_{\rho_2} \dots (e_{A_k})_{\mu_k} e_{A_k}^{\rho_k} \nabla_{\rho_k} \Psi_{\mu} \right] \\
&= O^{\mu_2} \dots O^{\mu_k} \left[\sum_{A_1} \sum_{A_2} \dots \sum_{A_k} \mathbf{g}(O, D_S^p e_{A_1}) \nabla_{A_1}(e_{A_2})_{\mu_2} \nabla_{A_2} \dots (e_{A_k})_{\mu_k} \nabla_{A_k} \Psi_{\mu} \right] \\
&+ O^{\mu_2} \dots O^{\mu_k} \left[\sum_{A_1} \sum_{A_2} \dots \sum_{A_k} \mathbf{g}(O, e_{A_1}) \mathbf{g}(D_S^p e_{A_1}, e_D) \nabla_D(e_{A_2})_{\mu_2} \nabla_{A_2} \dots (e_{A_k})_{\mu_k} \nabla_{A_k} \Psi_{\mu} \right] \\
&+ O^{\mu_2} \dots O^{\mu_k} \left[\sum_{A_1} \sum_{A_2} \dots \sum_{A_k} \mathbf{g}(O, e_{A_1}) D_S^p \nabla_{A_1}(e_{A_2})_{\mu_2} \nabla_{A_2} \dots (e_{A_k})_{\mu_k} \nabla_{A_k} \Psi_{\mu} \right] \quad (15.123)
\end{aligned}$$

We have,

$$\begin{aligned}
&\sup_{\mu} |O^{\mu_1} O^{\mu_2} \dots O^{\mu_k} \nabla_{\mu_1} D_S^p \nabla_{\mu_2} \dots \nabla_{\mu_k} \Psi_{\mu}| \\
&\leq |O|^{k-1} \left(\sup_A |\mathbf{g}(O, D_S^p e_A)| + \sup_{A,D} |\mathbf{g}(O, e_A)| |\mathbf{g}(D_S^p e_A, e_D)| \right) 2^k \sup_{\mu} \sup_{A_2, A_3, \dots, A_k} |\nabla_{A_1}(e_{A_2})_{\mu_2} \nabla_{A_2} \dots (e_{A_k})_{\mu_k} \nabla_{A_k} \Psi_{\mu}| \\
&+ |O|^{k-1} \left(\sup_A |\mathbf{g}(O, e_A)| \right) 2^k \sup_{\mu} \sup_{A_2, A_3, \dots, A_k} |D_S^p \nabla_{A_1}(e_{A_2})_{\mu_2} \nabla_{A_2} \dots (e_{A_k})_{\mu_k} \nabla_{A_k} \Psi_{\mu}| \\
&\leq C_1^k \sup_{\mu} \sup_{A_2, A_3, \dots, A_k; \mu_2, \mu_3, \dots, \mu_k} [|\nabla_{A_1}(e_{A_2})_{\mu_2} \nabla_{A_2} \dots (e_{A_k})_{\mu_k} \nabla_{A_k} \Psi_{\mu}| + |D_S^p \nabla_{A_1}(e_{A_2})_{\mu_2} \nabla_{A_2} \dots (e_{A_k})_{\mu_k} \nabla_{A_k} \Psi_{\mu}|] \quad (15.124)
\end{aligned}$$

We are left with estimating ¹³²

$$|D_S^p \nabla_{A_1}(e_{A_2})_{\mu_2} \nabla_{A_2} \dots (e_{A_k})_{\mu_k} \nabla_{A_k} \Psi_{\mu}| ,$$

This is a simple repetition with obvious modifications of what already done in the proof of Lemma 10.4 and we do not repeat here. The remaining part of the proof mimics the one of this lemma and we do not repeat it here.

15.7 Proof of lemma 4.4

We recall Lemma 4.4:

Corollary 4.4 *Under the assumptions of Lemma 4.3 the following estimates hold, with $p + k = J$:*

$$\begin{aligned}
&||\lambda|^{\phi(\Psi)} r^{\phi(\Psi) + (J-1) - \frac{2}{p}} D_{\nu}^p \nabla^k \Psi|_{L^p(S)} \leq \tilde{C}^{(1)} e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^{\alpha}} \frac{1}{\rho_2^p \rho_1^{k+1}} \\
&||\lambda|^{\phi(\Psi) + p} r^{\phi(\Psi) + k - \frac{2}{p}} D_{\lambda}^p \nabla^k \Psi|_{L^p(S)} \leq \tilde{C}^{(1)} e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{J!}{J^{\alpha}} \frac{1}{\rho_2^p \rho_1^{k+1}} .
\end{aligned}$$

With suitable $\rho_2 < \rho_1$.

¹³²notice that in order to estimate $D_S^p \nabla_{A_1}(e_{A_2})_{\mu_2} \nabla_{A_2} \dots (e_{A_k})_{\mu_k} \nabla_{A_k} \Psi_{\mu}$ we need to exploit the estimates of $D_S^p \nabla^Q \mathcal{O}$ which we have already obtained in subsection ??

Remark:

In the following proof the term $r^{\phi(\Psi)}$ has been neglected but in fact should be reinstated.

Proof: The proof is a consequence of the following lemma,

Lemma 15.2 *Defining $Q \geq (2|\Omega|_{\infty,S} + 1 + \sigma|X|_{\infty,S} + O(\epsilon))$, then the following estimates hold,*

$$||\underline{u}|^k |u|^l r^{-\frac{2}{p}} D_S^{p-l} D_\lambda^l \nabla^k \Psi|_{p,S} \leq C^{(1)} e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} Q^l \frac{1}{\rho_1^{p+k+1}}. \quad (15.125)$$

Proof: we have to prove the estimates by induction starting from $l = 0$; recall that from Lemma 10.7, omitting the factor $r^{-\frac{2}{p}}$ always present, we have, with $p + k = J - 1$

$$|\underline{u}^k D_S^p \nabla^k \Psi|_{L^p(S)} \leq C^{(1)} e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{1}{\rho_1^J} \quad (15.126)$$

We show that the estimates hold for $l = 1$, in fact, if $\Psi = \underline{\alpha}$, omitting lower order terms, it follows, recalling that

$$D_\lambda = \Omega D_3 + X \nabla ; \quad \Omega D_S = \frac{1}{2}(\underline{u} D_\nu + u D_\lambda) - \frac{u}{2} X \nabla$$

$$\begin{aligned} & |\underline{u}^k |u| D_S^{p-1} D_\lambda \nabla^k \underline{\alpha}|_{L^p(S)} \leq |\underline{u}^k D_S^{p-1} (2\Omega D_S + \underline{u} D_\nu + u X \nabla) \nabla^k \underline{\alpha}|_{L^p(S)} \\ & \leq 2|\underline{u}^k D_S^{p-1} \Omega D_S \nabla^k \underline{\alpha}|_{L^p(S)} + |\underline{u}^k D_S^{p-1} \underline{u} D_\nu \nabla^k \underline{\alpha}|_{L^p(S)} + |\underline{u}^k D_S^{p-1} u X \nabla^{k+1} \underline{\alpha}|_{L^p(S)} \\ & \leq \sum_{l=0}^{p-1} \binom{p-1}{l} |\underline{u}^k (D_S^l \Omega) D_S^{p-1-l} \nabla^k \underline{\alpha}|_{L^p(S)} + \sum_{l=0}^{p-1} \binom{p-1}{l} |\underline{u}^k (D_S^l \underline{u}) D_S^{p-1-l} D_\nu \nabla^k \underline{\alpha}|_{L^p(S)} \\ & + \sum_{l=0}^{p-1} \binom{p-1}{l} |\underline{u}^k (D_S^l u X) D_S^{p-1-l} \nabla^{k+1} \underline{\alpha}|_{L^p(S)} \\ & \leq 2|\Omega \underline{u}^k D_S^p \nabla^k \underline{\alpha}|_{L^p(S)} + |\underline{u}^{k+1} D_S^{p-1} D_\nu \nabla^k \underline{\alpha}|_{L^p(S)} + |\underline{u}^k u X D_S^{p-1} \nabla^{k+1} \underline{\alpha}|_{L^p(S)} \\ & + \left\{ \sum_{l=1}^{p-1} \binom{p-1}{l} |\underline{u}^k (D_S^l \Omega) D_S^{p-1-l} \nabla^k \underline{\alpha}|_{L^p(S)} + \sum_{l=1}^{p-1} \binom{p-1}{l} |\underline{u}^k (D_S^l \underline{u}) D_S^{p-1-l} D_\nu \nabla^k \underline{\alpha}|_{L^p(S)} \right. \\ & \left. + \sum_{l=1}^{p-1} \binom{p-1}{l} |\underline{u}^k (D_S^l u X) D_S^{p-1-l} \nabla^{k+1} \underline{\alpha}|_{L^p(S)} \right\}. \quad (15.127) \end{aligned}$$

We write in a symbolic way,

$$D_\nu \nabla^k \alpha = \nabla^{k+1} \alpha + \{l.o.t.\} \quad (15.128)$$

and we will omit for the moment the contribution due to the $\{l.o.t.\}$ part. Moreover we write approximately,

$$D_S \underline{u} = \frac{1}{2}(u\partial_3 + \underline{u}\partial_4)u = \frac{1}{2\Omega}(u\partial_u + \underline{u}\partial_{\underline{u}})\underline{u} + O(\nabla \underline{u}) = \frac{1}{2\Omega}(u\partial_u + \underline{u}\partial_{\underline{u}})\underline{u} = \frac{\underline{u}}{2\Omega} = \frac{\underline{u}}{4}$$

and

$$D_S^l \underline{u} = \frac{\underline{u}}{2^{2l}}. \quad (15.129)$$

Moreover

$$D_S \Omega = \frac{1}{2}(u\partial_3 + \underline{u}\partial_4)\Omega = \frac{\Omega}{2}(u\partial_3 + \underline{u}\partial_4) \log \Omega = -\Omega(\underline{u}\omega + u\underline{\omega}) \equiv \mathcal{O} \quad (15.130)$$

and we recall that we have the estimates for $D_S^j \mathcal{O}$ as we control $D_\nu^j \omega$, $D_\nu^j \underline{\omega}$, $D_\lambda^j \omega$, $D_\lambda^j \underline{\omega}$. Therefore we have

$$\begin{aligned} \sum_{l=1}^{p-1} \binom{p-1}{l} |\underline{u}^k (D_S^l \underline{u}) D_S^{p-1-l} D_\nu \nabla^k \underline{\omega}|_{L^p(S)} &\leq \sum_{l=1}^{p-1} \binom{p-1}{l} \frac{1}{2^{2l}} |\underline{u}^{k+1} D_S^{p-1-l} \nabla^{k+1} \underline{\omega}|_{L^p(S)} \\ &\leq C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \sum_{l=1}^{p-1} \binom{p-1}{l} \frac{1}{2^{2l}} \frac{1}{\rho_1^{p-1-l} \rho_1^{k+2}} \leq O(\rho_1) C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{1}{\rho_1^p \rho_1^{k+1}} \end{aligned} \quad (15.131)$$

$$\begin{aligned} \sum_{l=1}^{p-1} \binom{p-1}{l} |\underline{u}^k (D_S^l \Omega) D_S^{p-l} \nabla^k \underline{\omega}|_{L^p(S)} &\leq \sum_{l=1}^{p-1} \binom{p-1}{l} |(D_S^l \Omega)|_{\infty, S} |\underline{u}^k D_S^{p-l} \nabla^k \underline{\omega}|_{L^p(S)} \\ \sum_{l=1}^{p-1} \binom{p-1}{l} |(D_S^l \Omega)|_{\infty, S} C_4 e^{(p-l+k-2)(\delta+\Gamma(\lambda))} \frac{(p-l+k+1)!}{(p-l+k+1)^\alpha} \frac{1}{\rho_1^{p-l+k+1}} &\quad (15.132) \end{aligned}$$

Moreover ¹³³

$$|D_S^l \Omega|_{\infty, S} = |D_S^{l-1} \mathcal{O}|_{\infty, S} \leq c e^{(l-1)(\delta+\Gamma(\lambda))} \frac{(l+1)!}{(l+1)^\alpha} \frac{1}{\rho_1^{l+1}} \quad (15.133)$$

Remark:

Notice that in the estimate 15.133 turns out that in the denominator there is ρ_1 and not ρ ; this follows as we already know that the norms of $\mathcal{D}_S \Psi$ have ρ_1 , see Lemma 15.2, this implies the the estimates of the \mathcal{D}_S of the connection coefficients $\mathcal{D}_S^J \mathcal{O}$ have the same ρ_1 .

Therefore,

$$\sum_{l=1}^{p-1} \binom{p-1}{l} |\underline{u}^k (D_S^l \Omega) D_S^{p-l} \nabla^k \underline{\omega}|_{L^p(S)}$$

¹³³The factor $(l+1)$ instead of l is due to the delicate estimates of $D_\nu^j \omega$ and of $D_\lambda^j \underline{\omega}$.

$$\begin{aligned}
&\leq \sum_{l=1}^{p-1} \binom{p-1}{l} ce^{(l-1)(\delta+\Gamma(\lambda))} \frac{(l+1)!}{(l+1)^\alpha} \frac{1}{\rho^{l+1}} C_4 e^{(p-l+k-2)(\delta+\Gamma(\lambda))} \frac{(p-l+k+1)!}{(p-l+k+1)^\alpha} \frac{1}{\rho_1^{p-l+k+1}} \\
&\leq C_4 e^{(p+k+1-2)(\delta+\Gamma(\lambda))} \frac{(p+k+1)!}{(p+k+1)^\alpha} \frac{1}{\rho_1^l \rho_1^{p-l+k+1}} \left[\frac{(p+k+1)^\alpha}{(p+k+1)!} \frac{ce^{-\delta}}{\rho_1} \sum_{l=1}^{p-1} \binom{p-1}{l} \left(\frac{\rho_1}{\rho}\right)^{l+1} \frac{(l+1)!}{(l+1)^\alpha} \frac{(p-l+k+1)!}{(p-l+k+1)^\alpha} \right] \\
&\leq C_4 e^{(p+k+1-2)(\delta+\Gamma(\lambda))} \frac{(p+k+1)!}{(p+k+1)^\alpha} \frac{1}{\rho_1^{p+k+1}} \left[\frac{(p+k+1)^\alpha}{(p+k+1)!} \frac{ce^{-\delta}}{\rho_1} \sum_{l=1}^{p-1} \left(\frac{\rho_1}{\rho}\right)^{l+1} \frac{(p-1)!}{(p-1-l)!} \frac{(l+1)!}{(l+1)^\alpha} \frac{(p-l+k+1)!}{(p-l+k+1)^\alpha} \right] \\
&\leq \left(C_4 e^{(p+k+1-2)(\delta+\Gamma(\lambda))} \frac{(p+k+1)!}{(p+k+1)^\alpha} \frac{1}{\rho_1^{p+k+1}} \right) \left[\frac{ce^{-\delta}}{\rho_1} \sum_{l=1}^{p-1} \left(\frac{\rho_1}{\rho_{0,10}}\right)^{l+1} \frac{1}{(l+1)^{\alpha-1}} \frac{(p+k+1)^\alpha}{(p-l+k+1)^\alpha} \frac{(p-1)!}{(p-1-l)!} \frac{(p-l+k+1)!}{(p+k+1)!} \right]
\end{aligned} \tag{15.134}$$

Remark:

The result is that the sum over l is bounded and the factor in front $\frac{ce^{-\delta}}{\rho_1}$ guarantees it is a small correction choosing δ sufficiently large.

More precisely

$$\begin{aligned}
&\sum_{l=1}^{\frac{p}{2}} \left(\frac{\rho_1}{\rho}\right)^{l+1} \frac{1}{(l+1)^{\alpha-1}} \frac{(p+k+1)^\alpha}{(p-l+k+1)^\alpha} \frac{(p-1)!}{(p-1-l)!} \frac{(p-l+k+1)!}{(p+k+1)!} \\
&\leq \left(\frac{\rho_1}{\rho}\right) 2^\alpha \sum_{l=1}^{\frac{p}{2}} \frac{1}{(l+1)^{\alpha-1}} \left[\frac{(p-1)!}{(p+k+1)!} \frac{(p-l+k+1)!}{(p-1-l)!} \right] \leq c \left(\frac{\rho_1}{\rho}\right) 2^\alpha \sum_{l=1}^{\frac{p}{2}} \frac{1}{(l+1)^{\alpha-1}} \\
&\leq \left(\frac{\rho_1}{\rho}\right) 2^\alpha c_1 \leq C_1 .
\end{aligned} \tag{15.135}$$

$$\begin{aligned}
&\sum_{l=\frac{p}{2}+1}^p \left(\frac{\rho_1}{\rho}\right)^{l+1} \frac{1}{(l+1)^{\alpha-1}} \frac{(p+k+1)^\alpha}{(p-l+k+1)^\alpha} \frac{(p-1)!}{(p-1-l)!} \frac{(p-l+k+1)!}{(p+k+1)!} \\
&\leq \left(\frac{\rho_1}{\rho}\right)^{\frac{p}{2}} p^\alpha \sum_{l=\frac{p}{2}+1}^p \frac{1}{(l+1)^{\alpha-1}} \leq C_2 .
\end{aligned} \tag{15.136}$$

The estimate of the third sum is done exactly in the same way and we do not repeat it here. The final result is that

$$\begin{aligned}
&\left\{ \sum_{l=1}^{p-1} \binom{p-1}{l} |\underline{u}^k(D_S^l \Omega) D_S^{p-l} \nabla^k \underline{\alpha}|_{L^p(S)} + \sum_{l=1}^{p-1} \binom{p-1}{l} |\underline{u}^k(D_S^l \underline{u}) D_S^{p-1-l} D_\nu \nabla^k \underline{\alpha}|_{L^p(S)} \right. \\
&\quad \left. + \sum_{l=1}^{p-1} \binom{p-1}{l} |\underline{u}^k(D_S^l u X) D_S^{p-1-l} D_\nu \nabla^{k+1} \underline{\alpha}|_{L^p(S)} \right\} \leq O(\epsilon) C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{1}{\rho_1^p \rho_1^{k+1}}
\end{aligned} \tag{15.137}$$

Therefore we have

$$|\underline{u}^k|u|D_S^{p-1} D_\lambda \nabla^k \underline{\alpha}|_{L^p(S)}$$

$$\begin{aligned}
&\leq 2|\Omega \underline{u}^k D_S^p \nabla^k \underline{\alpha}|_{L^p(S)} + |\underline{u}^{k+1} D_S^{p-1} \nabla^{k+1} \underline{\alpha}|_{L^p(S)} + \\
&\quad |\underline{u}^k u X D_S^{p-1} \nabla^{k+1} \underline{\alpha}|_{L^p(S)} + O(\epsilon) C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{1}{\rho_1^p \rho_1^{k+1}} \\
&\leq 2|\Omega|_{\infty, S} C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda, \nu)} \rho_1^p \rho_1^{k+1} + C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{1}{\rho_1^{p-1} \rho_1^{k+2}} \\
&\quad + |X|_{\infty, S} \frac{|u|}{|\underline{u}|} C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{1}{\rho_1^{p-1} \rho_1^{k+2}} + O(\epsilon) C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{1}{\rho_1^p \rho_1^{k+1}} \\
&\leq C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{(2|\Omega|_{\infty, S} + 1 + \sigma|X|_{\infty, S} + O(\epsilon))}{\rho_1^p \rho_1^{k+1}} \leq \\
&\quad C^{(1)} e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \left(\frac{Q\rho_3}{\rho_1} \right) \frac{1}{\rho_1^{p-1} \rho_3 \rho_1^{k+1}}. \tag{15.138}
\end{aligned}$$

Where ρ_3 plays no substantial role and could be omitted, we introduced it only for the sake of simplicity in the calculation

$$Q \geq (2|\Omega|_{\infty, S} + 1 + \sigma|X|_{\infty, S} + O(\epsilon)). \tag{15.139}$$

If $\Psi \neq \underline{\alpha}$, omitting lower order terms, where $\tilde{\Psi}$ can be $\underline{\alpha}$,

$$\begin{aligned}
&|\underline{u}^k |u| D_S^{p-1} D_\lambda \nabla^k \Psi|_{L^p(S)} = \frac{|u|}{|\underline{u}|} |\underline{u}^{k+1} D_S^{p-1} \nabla^{k+1} \tilde{\Psi}|_{L^p(S)} \\
&\leq \frac{|u|}{|\underline{u}|} C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{1}{\rho_1^{p-1} \rho_1^{k+2}} \leq \sigma \frac{\rho_3}{\rho_1} C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{1}{\rho_1^{p-1} \rho_3 \rho_1^{k+1}} \\
&\quad . \tag{15.140}
\end{aligned}$$

Therefore we assume,

$$|\underline{u}^k |u|^{l-1} D_S^{p-(l-1)} D_\lambda^{l-1} \nabla^k \Psi|_{L^p(S)} \leq \left(\frac{Q\rho_3}{\rho_1} \right)^{l-1} C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{1}{\rho_1^{p-(l-1)} \rho_3^{l-1} \rho_1^{k+1}} \tag{15.141}$$

and we want to prove the estimate for $|\underline{u}^k |u|^l D_S^{p-l} D_\lambda^l \nabla^k \Psi|_{L^p(S)}$ namely

$$|\underline{u}^k |u|^l D_S^{p-l} D_\lambda^l \nabla^k \Psi|_{L^p(S)} \leq \left(\frac{Q\rho_3}{\rho_1} \right)^l C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{1}{\rho_1^{p-l} \rho_3^l \rho_1^{k+1}} \tag{15.142}$$

Proof: Let $\Psi = \underline{\alpha}$ then

$$\begin{aligned}
&|\underline{u}^k |u|^l D_S^{p-l} D_\lambda^l \nabla^k \underline{\alpha}|_{L^p(S)} \leq |\underline{u}^k |u|^{l-1} D_S^{p-l} (2\Omega D_S + \underline{u} D_\nu + u X \nabla) D_\lambda^{l-1} \nabla^k \underline{\alpha}|_{L^p(S)} \\
&\leq 2|\underline{u}^k |u|^{l-1} D_S^{p-l} \Omega D_S D_\lambda^{l-1} \nabla^k \underline{\alpha}|_{L^p(S)} + |\underline{u}^k |u|^{l-1} D_S^{p-l} \underline{u} D_\nu D_\lambda^{l-1} \nabla^k \underline{\alpha}|_{L^p(S)} + |\underline{u}^k D_S^{p-l} u X \nabla D_\lambda^{l-1} \nabla^k \underline{\alpha}|_{L^p(S)}
\end{aligned}$$

$$\begin{aligned}
&\leq 2 \sum_{s=0}^{p-l} \binom{p-l}{s} |\underline{u}^k|u|^{l-1} (D_S^s \Omega) D_S^{p-(l-1)-s} D_\lambda^{l-1} \nabla^k \underline{\alpha}|_{L^p(S)} + \sum_{s=0}^{p-l} \binom{p-l}{s} |\underline{u}^k|u|^{l-1} (D_S^s \underline{u}) D_S^{p-l-s} D_\nu D_\lambda^{l-1} \nabla^k \underline{\alpha}|_{L^p(S)} \\
&\quad + \sum_{s=0}^{p-l} \binom{p-l}{s} |\underline{u}^k|u|^{l-1} (D_S^s u X) D_S^{p-l-s} \nabla D_\lambda^{l-1} \nabla^k \underline{\alpha}|_{L^p(S)} \\
&\leq 2 |\Omega \underline{u}^k|u|^{l-1} D_S^{p-(l-1)} D_\lambda^{l-1} \nabla^k \underline{\alpha}|_{L^p(S)} + |\underline{u}^{k+1}|u|^{l-1} D_S^{p-l} D_\nu D_\lambda^{l-1} \nabla^k \underline{\alpha}|_{L^p(S)} + |\underline{u}^k|u|^l X D_S^{p-l} \nabla D_\lambda^{l-1} \nabla^k \underline{\alpha}|_{L^p(S)} \\
&\quad + \left\{ 2 \sum_{s=1}^{p-l} \binom{p-l}{s} |\underline{u}^k|u|^{l-1} (D_S^s \Omega) D_S^{p-(l-1)-s} D_\lambda^{l-1} \nabla^k \underline{\alpha}|_{L^p(S)} + \sum_{s=1}^{p-l} \binom{p-l}{s} |\underline{u}^k|u|^{l-1} (D_S^s \underline{u}) D_S^{p-l-s} D_\nu D_\lambda^{l-1} \nabla^k \underline{\alpha}|_{L^p(S)} \right. \\
&\quad \left. + \sum_{s=1}^{p-l} \binom{p-l}{s} |\underline{u}^k|u|^{l-1} (D_S^s u X) D_S^{p-l-s} \nabla D_\lambda^{l-1} \nabla^k \underline{\alpha}|_{L^p(S)} \right\} \\
&\leq 2 |\Omega \underline{u}^k|u|^{l-1} D_S^{p-(l-1)} D_\lambda^{l-1} \nabla^k \underline{\alpha}|_{L^p(S)} + |\underline{u}^{k+1}|u|^{l-1} D_S^{p-l} D_\nu D_\lambda^{l-1} \nabla^k \underline{\alpha}|_{L^p(S)} + |\underline{u}^k|u|^l X D_S^{p-l} \nabla D_\lambda^{l-1} \nabla^k \underline{\alpha}|_{L^p(S)} \\
&\quad + \{l.o.t.(I)\} \\
&\leq 2 |\Omega \underline{u}^k|u|^{l-1} D_S^{p-(l-1)} D_\lambda^{l-1} \nabla^k \underline{\alpha}|_{L^p(S)} + |\underline{u}^{k+1}|u|^{l-1} D_S^{p-l} D_\lambda^{l-1} D_\nu \nabla^k \underline{\alpha}|_{L^p(S)} + |\underline{u}^k|u|^l X D_S^{p-l} D_\lambda^{l-1} \nabla^{k+1} \underline{\alpha}|_{L^p(S)} \\
&\quad + \left\{ |\underline{u}^{k+1}|u|^{l-1} D_S^{p-l} [D_\nu, D_\lambda^{l-1}] \nabla^k \underline{\alpha}|_{L^p(S)} + |\underline{u}^k|u|^l X D_S^{p-l} [\nabla, D_\lambda^{l-1}] \nabla^k \underline{\alpha}|_{L^p(S)} \right\} + \left\{ l.o.t.(I) \right\} \quad (15.143) \\
&\leq 2 |\Omega \underline{u}^k|u|^{l-1} D_S^{p-(l-1)} D_\lambda^{l-1} \nabla^k \underline{\alpha}|_{L^p(S)} + |\underline{u}^{k+1}|u|^{l-1} D_S^{p-l} D_\lambda^{l-1} D_\nu \nabla^k \underline{\alpha}|_{L^p(S)} + |\underline{u}^k|u|^l X D_S^{p-l} D_\lambda^{l-1} \nabla^{k+1} \underline{\alpha}|_{L^p(S)} \\
&\quad + \left\{ l.o.t.(II) \right\} + \left\{ l.o.t.(I) \right\}
\end{aligned}$$

where

$$\left\{ l.o.t.(II) \right\} = \left\{ |\underline{u}^{k+1}|u|^{l-1} D_S^{p-l} [D_\nu, D_\lambda^{l-1}] \nabla^k \underline{\alpha}|_{L^p(S)} + |\underline{u}^k|u|^l X D_S^{p-l} [\nabla, D_\lambda^{l-1}] \nabla^k \underline{\alpha}|_{L^p(S)} \right\} . \quad (15.144)$$

Let us examine in more detail $\left\{ l.o.t.(II) \right\}$, we have for the first term,

$$|\underline{u}^{k+1}|u|^{l-1} D_S^{p-l} [D_\nu, D_\lambda^{l-1}] \nabla^k \underline{\alpha}|_{L^p(S)} \leq \sum_{s=0}^{l-2} |\underline{u}^{k+1}|u|^{l-1} D_S^{p-l} D_\lambda^{l-2-s} [D_\nu, D_\lambda] D_\lambda^s \nabla^k \underline{\alpha}|_{L^p(S)} ,$$

$D_\lambda^s \nabla^k \underline{\alpha}$ is a $(s+k+2)$ tensor, therefore we have

$$\begin{aligned}
[D_\nu, D_\lambda] D_\lambda^s \nabla^k \underline{\alpha}_{\tau_1 \dots \tau_{k+2}} &= \sum_{j=1}^{s+k+2} R^{\tilde{\tau} j} (e_\lambda, e_\nu)_{\tau_j} (D^s \nabla^k \underline{\alpha})_{\tau_1 \dots \tilde{\tau}_j \dots \tau_{s+k+2}} \\
&\leq (s+k+2) R D_\lambda^{s-1} D \nabla^k \underline{\alpha} \quad (15.145)
\end{aligned}$$

where D could be D_λ, D_ν, ∇ and we can move, apart lower order terms on the r.h.s. near ∇^k . Therefore

$$|\underline{u}^{k+1}|u|^{l-1} D_S^{p-l} [D_\nu, D_\lambda^{l-1}] \nabla^k \underline{\alpha}|_{L^p(S)} \leq \sum_{s=0}^{l-2} (s+k+2) |\underline{u}^{k+1}|u|^{l-1} D_S^{p-l} D_\lambda^{l-2-s} R D_\lambda^{s-1} D \nabla^k \underline{\alpha}|_{L^p(S)}$$

$$\begin{aligned}
&\leq \sum_{s=0}^{l-2} (s+k+2) \sum_{t=0}^{l-2-s} \binom{l-2-s}{t} |\underline{u}^{k+1}| u^{l-1} D_S^{p-l} (D_\lambda^t R) D_\lambda^{l-2-s-t} D_\lambda^{s-1} D \nabla^k \underline{\alpha}|_{L^p(S)} \quad (15.146) \\
&\leq \sum_{s=0}^{l-2} (s+k+2) \sum_{t=0}^{l-2-s} \sum_{q=0}^{p-l} \binom{l-2-s}{t} \binom{p-l}{q} |\underline{u}^{k+1}| u^{l-1} (D_S^q D_\lambda^t R) D_S^{p-l-q} D_\lambda^{l-3-t} D \nabla^k \underline{\alpha}|_{L^p(S)}
\end{aligned}$$

Observe that there are $p+k-2$ derivatives and as we have to introduce a $|\cdot|_{\infty, S}$ norm the number of derivatives turns out to be $p+k-1$. Let us try a rough estimate, assuming $D = D_\lambda$,

$$\begin{aligned}
&|\underline{u}^{k+1}| u^{l-1} D_S^{p-l} [D_\nu, D_\lambda^{l-1}] \nabla^k \underline{\alpha}|_{L^p(S)} \\
&\leq \sum_{s=0}^{l-2} (s+k+2) \sum_{t=0}^{l-2-s} \sum_{q=0}^{p-l} \binom{l-2-s}{t} \binom{p-l}{q} \|u\|^t (D_S^q D_\lambda^t R)|_{\infty, S} |\underline{u}^{k+1}| u^{l-t-1} D_S^{p-l-q} D_\lambda^{l-2-t} \nabla^k \underline{\alpha}|_{L^p(S)} \\
&\leq \sum_{s=0}^{l-2} (s+k+2) \sum_{t=0}^{l-2-s} \sum_{q=0}^{p-l} \binom{l-2-s}{t} \binom{p-l}{q} \left(\frac{Q\rho_3}{\rho_1}\right)^t \frac{1}{\rho_1^{q+1} \rho_3^t} e^{(q+t)(\delta+\Gamma(\lambda))} \frac{(q+t+2)!}{(q+t+2)^a} \cdot \\
&\quad \cdot \left(\frac{Q\rho_3}{\rho_1}\right)^{l-2-t} \frac{1}{\rho_1^{p-l-q} \rho_3^{l-2-t} \rho_1^{k+1}} e^{(p+k-q-t-3)(\delta+\Gamma(\lambda))} \frac{(p+k-q-t-1)!}{(p+k-q-t-1)^\alpha} \\
&\leq e^{(p+k-3)(\delta+\Gamma)} \left(\frac{Q\rho_3}{\rho_1}\right)^{l-2} \frac{1}{\rho_1^{p-l+1} \rho_3^{l-2} \rho_1^{k+1}} \left[\sum_{s=0}^{l-2} (s+k+2) \sum_{t=0}^{l-2-s} \sum_{q=0}^{p-l} \binom{l-2-s}{t} \binom{p-l}{q} \frac{(q+t+2)!}{(q+t+2)^a} \frac{(p+k-q-t-1)!}{(p+k-q-t-1)^\alpha} \right] \\
&\leq e^{(p+k-1)(\delta+\Gamma)} \left(\frac{\rho_1 e^{-2(\delta+\Gamma)}}{Q^2}\right) \left(\frac{Q\rho_3}{\rho_1}\right)^l \frac{1}{\rho_1^{p-l} \rho_3^l \rho_1^{k+1}} \\
&\quad \left[\sum_{s=0}^{l-2} (s+k+2) \sum_{t=0}^{l-2-s} \sum_{q=0}^{p-l} \binom{l-2-s}{t} \binom{p-l}{q} \frac{(q+t+2)!}{(q+t+2)^a} \frac{(p+k-q-t-1)!}{(p+k-q-t-1)^\alpha} \right] \\
&\leq \left(\frac{\rho_1 e^{-2(\delta+\Gamma)}}{Q^2}\right) \left(\left(\frac{Q\rho_3}{\rho_1}\right)^l \frac{1}{\rho_1^{p-l} \rho_3^l \rho_1^{k+1}}\right) \left(e^{(p+k-1)(\delta+\Gamma(\lambda))} \frac{(p+k+1)!}{(p+k+1)^\alpha}\right) \cdot \\
&\quad \cdot \frac{(p+k+1)^\alpha}{(p+k+1)!} \left[\sum_{s=0}^{l-2} (s+k+2) \sum_{t=0}^{l-2-s} \sum_{q=0}^{p-l} \binom{l-2-s}{t} \binom{p-l}{q} \frac{(q+t+2)!}{(q+t+2)^a} \frac{(p+k-q-t-1)!}{(p+k-q-t-1)^\alpha} \right] \\
&\leq \left(\frac{\rho_1 e^{-2(\delta+\Gamma)}}{Q^2}\right) \left(C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \left(\frac{Q\rho_3}{\rho_1}\right)^l \frac{1}{\rho_1^{p-l} \rho_3^l \rho_1^{k+1}}\right) \cdot \\
&\quad \cdot \frac{(p+k+1)^\alpha}{(p+k+1)!} \left[\sum_{s=0}^{l-2} (s+k+2) \sum_{t=0}^{l-2-s} \sum_{q=0}^{p-l} \binom{l-2-s}{t} \binom{p-l}{q} \frac{(q+t+2)!}{(q+t+2)^a} \frac{(p+k-q-t-1)!}{(p+k-q-t-1)^\alpha} \right]
\end{aligned} \tag{15.147}$$

We have

$$D^{p-2-s} AB = \sum_{h=0}^{p-2-s} \binom{p-2-s}{h} (D^h A) D^{p-2-s-h} B$$

$$\begin{aligned}
&= D^{p-l} D^{l-2-s} AB = D^{p-l} \sum_{t=0}^{l-2-s} \binom{l-2-s}{t} (D^t A) D^{l-2-s-t} B \\
&= \sum_{q=0}^{p-l} \sum_{t=0}^{l-2-s} \binom{p-l}{q} \binom{l-2-s}{t} (D^{t+q} A) D^{p-2-s-(t+q)} B \\
&= \sum_{h=0}^{p-2-s} \left[\delta(h-(t+q)) \sum_{q=0}^{p-l} \sum_{t=0}^{l-2-s} \binom{p-l}{q} \binom{l-2-s}{t} \right] (D^h A) D^{p-2-s-h} B
\end{aligned}$$

Therefore

$$\left[\delta(h-(t+q)) \sum_{q=0}^{p-l} \sum_{t=0}^{l-2-s} \binom{p-l}{q} \binom{l-2-s}{t} \right] = \binom{p-2-s}{h} . \quad (15.148)$$

and

$$\begin{aligned}
&\left[\sum_{s=0}^{l-2} (s+k+2) \sum_{t=0}^{l-2-s} \sum_{q=0}^{p-l} \binom{l-2-s}{t} \binom{p-l}{q} \frac{(q+t+2)!}{(q+t+2)^a} \frac{(p+k-q-t-1)!}{(p+k-q-t-1)^\alpha} \right] \\
&= \sum_{s=0}^{l-2} (s+k+2) \sum_{h=0}^{p-2-s} \binom{p-2-s}{h} \frac{(h+2)!}{(h+2)^a} \frac{(p+k-h-1)!}{(p+k-h-1)^\alpha} \quad (15.149)
\end{aligned}$$

and

$$\begin{aligned}
&|\underline{u}^{k+1} |u|^{l-1} D_S^{p-l} [D_\nu, D_\lambda^{l-1}] \nabla^k \underline{\alpha}|_{L^p(S)} \leq \\
&\leq \left(\frac{\rho_1 e^{-2(\delta+\Gamma)}}{Q^2} \right) \left(C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \left(\frac{Q\rho_3}{\rho_1} \right)^l \frac{1}{\rho_1^{p-l} \rho_3^l \rho_1^{k+1}} \right) \cdot \\
&\cdot \frac{(p+k+1)^\alpha}{(p+k+1)!} \left[\sum_{s=0}^{l-2} (s+k+2) \sum_{h=0}^{p-2-s} \binom{p-2-s}{h} \frac{(h+2)!}{(h+2)^a} \frac{(p+k-h-1)!}{(p+k-h-1)^\alpha} \right] \\
&\leq \left(\frac{\rho_1 e^{-2(\delta+\Gamma)}}{Q^2} \right) \left(C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \left(\frac{Q\rho_3}{\rho_1} \right)^l \frac{1}{\rho_1^{p-l} \rho_3^l \rho_1^{k+1}} \right) \cdot \\
&\cdot \left[\sum_{s=0}^{l-2} (s+k+2) \sum_{h=0}^{p-2-s} \frac{(p+k+1)^\alpha}{(h+2)^\alpha (p+k-h-1)^\alpha} \frac{(p-2-s)!}{(p-2-s-h)!} \frac{(h+2)!}{h!} \frac{(p+k-h-1)!}{(p+k+1)!} \right]
\end{aligned}$$

Let us look at the term in parenthesis considering the sum over h only up to $\frac{p}{2}$,

$$\begin{aligned}
&\left[\sum_{s=0}^{l-2} (s+k+2) \sum_{h=0}^{\frac{p}{2}} \frac{(p+k+1)^\alpha}{(h+2)^\alpha (p+k-h-1)^\alpha} \frac{(p-2-s)!}{(p-2-s-h)!} \frac{(h+2)!}{h!} \frac{(p+k-h-1)!}{(p+k+1)!} \right] \\
&\leq 2^\alpha \frac{l(l+k)}{(p+k)(p+k+1)} \sum_{h=0}^{\frac{p}{2}} \frac{1}{(h+2)^{\alpha-2}} \left[\frac{(p-l)!}{(p-l-h)!} \frac{(p+k-h-1)!}{(p+k-1)!} \right] \\
&\leq 2^\alpha c \sum_{h=0}^{\frac{p}{2}} \frac{1}{(h+2)^{\alpha-2}} \leq C , \quad (15.150)
\end{aligned}$$

recalling that $l \leq p$ and $\alpha > 3$. Assume now h between $\frac{p}{2} + 1$ and p then

$$\begin{aligned}
& \left[\sum_{s=0}^{l-2} (s+k+2) \sum_{h=\frac{p}{2}+1}^p \frac{(p+k+1)^\alpha}{(h+2)^\alpha (p+k-h-1)^\alpha} \frac{(p-2-s)!}{(p-2-s-h)!} \frac{(h+2)!}{h!} \frac{(p+k-h-1)!}{(p+k+1)!} \right] \\
& \leq l(l+k) \sum_{h=\frac{p}{2}+1}^p \frac{1}{(p+k-h-1)^\alpha} \frac{(p+k+1)^\alpha}{(h+2)^\alpha (p+k+1)(p+k)} \left[\frac{(p-2-s)!}{(p-2-s-h)!} \frac{(p+k-h-1)!}{(p+k-1)!} \right] \\
& \leq c \frac{l(l+k)}{(p+k+1)(p+k)} \sum_{h=\frac{p}{2}+1}^p \frac{1}{(p+k-h-1)^\alpha} \frac{(p+k+1)^\alpha}{(h+2)^\alpha} \leq c_1 \sum_{h=\frac{p}{2}+1}^p \frac{1}{(p+k-h-1)^\alpha} \frac{(p+k+1)^\alpha}{(h+2)^\alpha} \leq C.
\end{aligned}$$

In fact the last sum can easily be estimated in the following way, assume $p \geq k$,

$$\sum_{h=\frac{p}{2}+1}^p \frac{1}{(p+k-h-1)^\alpha} \frac{(p+k+1)^\alpha}{(h+2)^\alpha} \leq \frac{(2p)^\alpha}{\left(\frac{p}{2}\right)^\alpha} \sum_{h=\frac{p}{2}+1}^p \frac{1}{(p+k-h-1)^\alpha} \leq 2^{2\alpha} c.$$

Assume $p < k$ then

$$\sum_{h=\frac{p}{2}+1}^p \frac{1}{(p+k-h-1)^\alpha} \frac{(p+k+1)^\alpha}{(h+2)^\alpha} \leq \frac{(2k)^\alpha}{k^\alpha} \sum_{h=\frac{p}{2}+1}^p \frac{1}{(h+2)^\alpha} \leq 2^\alpha c_1. \quad (15.151)$$

Therefore our rough estimate suggests that

$$\begin{aligned}
& |\underline{u}^{k+1} |u|^{l-1} D_S^{p-l} [D_\nu, D_\lambda^{l-1}] \nabla^k \underline{\alpha}|_{L^p(S)} \leq \left(2^{2\alpha} \rho_1 e^{-2(\delta+\Gamma(\lambda))} \right) \\
& \quad \left(C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda, \nu)} \left(\frac{Q\rho_3}{\rho_1} \right)^l \frac{1}{\rho_1^{p-l} \rho_3^l \rho_1^{k+1}} \right) \\
& \leq \left(2^{2\alpha} \frac{\rho_1 e^{-2(\delta+\Gamma)}}{Q} \right) \left(C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda, \nu)} \left(\frac{Q\rho_3}{\rho_1} \right)^{l-1} \frac{1}{\rho_1^{p-(l-1)} \rho_3^{l-1} \rho_1^{k+1}} \right) \\
& \leq O(\epsilon) C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \left(\frac{Q\rho_3}{\rho_1} \right)^{l-1} \frac{1}{\rho_1^{p-(l-1)} \rho_3^{l-1} \rho_1^{k+1}}
\end{aligned}$$

and, more in general that,

$$\left\{ l.o.t.(II) \right\} \leq O(\epsilon) C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \left(\frac{Q\rho_3}{\rho_1} \right)^{l-1} \frac{1}{\rho_1^{p-(l-1)} \rho_3^{l-1} \rho_1^{k+1}} \quad (15.152)$$

The estimate of $\left\{ l.o.t.(I) \right\}$ goes as the one done in the case $l = 1$ and there is no need to repeat and again,

$$\left\{ l.o.t.(I) \right\} \leq O(\epsilon) C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \left(\frac{Q\rho_3}{\rho_1} \right)^{l-1} \frac{1}{\rho_1^{p-(l-1)} \rho_3^{l-1} \rho_1^{k+1}}. \quad (15.153)$$

Going back we have, therefore,

$$\begin{aligned}
& |\underline{u}^k |u|^l D_S^{p-l} D_\lambda^l \nabla^k \underline{\alpha}|_{L^p(S)} \\
& \leq 2|\Omega|_{\infty,S} |\underline{u}^k |u|^{l-1} D_S^{p-(l-1)} D_\lambda^{l-1} \nabla^k \underline{\alpha}|_{L^p(S)} + |\underline{u}^{k+1} |u|^{l-1} D_S^{p-l} D_\lambda^{l-1} D_\nu \nabla^k \underline{\alpha}|_{L^p(S)} + |\underline{u}^k |u|^l X D_S^{p-l} D_\lambda^{l-1} \nabla^{k+1} \underline{\alpha}|_{L^p(S)} \\
& + O(\epsilon) C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \left(\frac{Q\rho_3}{\rho_1} \right)^{l-1} \frac{1}{\rho_1^{p-(l-1)} \rho_3^{l-1} \rho_1^{k+1}} \\
& \leq 2|\Omega|_{\infty,S} |\underline{u}^k |u|^{l-1} D_S^{p-(l-1)} D_\lambda^{l-1} \nabla^k \underline{\alpha}|_{L^p(S)} + |\underline{u}^{k+1} |u|^{l-1} D_S^{p-l} D_\lambda^{l-1} \nabla^{k+1} \underline{\alpha}|_{L^p(S)} + \frac{|u|}{|\underline{u}|} |X|_{\infty,S} |\underline{u}^{k+1} |u|^{l-1} D_S^{p-l} D_\lambda^{l-1} \nabla^{k+1} \underline{\alpha}|_{L^p(S)} \\
& + O(\epsilon) C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \left(\frac{Q\rho_3}{\rho_1} \right)^{l-1} \frac{1}{\rho_1^{p-(l-1)} \rho_3^{l-1} \rho_1^{k+1}} \\
& \leq (2|\Omega|_{\infty,S} + 1 + \sigma |X|_{\infty,S}) \left(\frac{Q\rho_3}{\rho_1} \right)^{l-1} C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{1}{\rho_1^{p-(l-1)} \rho_3^{l-1} \rho_1^{k+1}} + \\
& \quad O(\epsilon) C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda,\nu)} \left(\frac{Q\rho_3}{\rho_1} \right)^{l-1} \frac{1}{\rho_1^{p-(l-1)} \rho_3^{l-1} \rho_1^{k+1}} \\
& \leq (2|\Omega|_{\infty,S} + 1 + \sigma |X|_{\infty,S} + O(\epsilon)) \frac{\rho_3}{\rho_1} \left(\frac{Q\rho_3}{\rho_1} \right)^{l-1} C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{1}{\rho_1^{p-l} \rho_3^l \rho_1^{k+1}} \\
& \leq C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \left(\frac{Q\rho_3}{\rho_1} \right)^l \frac{1}{\rho_1^{p-l} \rho_3^l \rho_1^{k+1}}. \tag{15.154}
\end{aligned}$$

Let $\Psi \neq \underline{\alpha}$ then

$$\begin{aligned}
& |\underline{u}^k |u|^l D_S^{p-l} D_3^l \nabla^k \Psi|_{L^p(S)} \leq |\underline{u}^k |u|^l D_S^{p-l} D_3^{l-1} \nabla^k D_3 \Psi|_{L^p(S)} + \left\{ |\underline{u}^k |u|^l D_S^{p-l} D_3^{l-1} [D_3, \nabla^k] \Psi|_{L^p(S)} \right\} \\
& \leq |\underline{u}^k |u|^l D_S^{p-l} D_3^{l-1} \nabla^{k+1} \tilde{\Psi}|_{L^p(S)} + \left\{ |\underline{u}^k |u|^l D_S^{p-l} D_3^{l-1} [D_3, \nabla^k] \Psi|_{L^p(S)} \right\} \\
& \leq \frac{|u|}{|\underline{u}|} |\underline{u}^{k+1} |u|^{l-1} D_S^{p-l} D_3^{l-1} \nabla^{k+1} \tilde{\Psi}|_{L^p(S)} + \left\{ |\underline{u}^k |u|^l D_S^{p-l} D_3^{l-1} [D_3, \nabla^k] \Psi|_{L^p(S)} \right\} \tag{15.155}
\end{aligned}$$

The term $\left\{ |\underline{u}^k |u|^l D_S^{p-l} D_3^{l-1} [D_3, \nabla^k] \Psi|_{L^p(S)} \right\}$ has to be estimated as before, therefore

$$\begin{aligned}
& |\underline{u}^k |u|^l D_S^{p-l} D_3^l \nabla^k \Psi|_{L^p(S)} \\
& \leq \sigma \left(\frac{3\rho_3}{\rho_1} \right)^{l-1} \frac{\rho_3}{\rho_1} C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{1}{\rho_1^{p-l} \rho_3^l \rho_1^{k+1}} + O(\epsilon) C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \\
& \quad \left(\frac{Q\rho_3}{\rho_1} \right)^{l-1} \frac{1}{\rho_1^{p-(l-1)} \rho_3^{l-1} \rho_1^{k+1}} \\
& \leq (\sigma + O(\epsilon)) \frac{\rho_3}{\rho_1} \left(\frac{Q\rho_3}{\rho_1} \right)^{l-1} C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{1}{\rho_1^{p-l} \rho_3^l \rho_1^{k+1}}
\end{aligned}$$

$$\leq C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \left(\frac{Q\rho_3}{\rho_1} \right)^l \frac{1}{\rho_1^{p-l} \rho_3^l \rho_1^{k+1}}.$$

Therefore, assuming the previous Proofs of lower order terms, we have proved for any Ψ and any l the estimate,

$$|\underline{u}^k |u|^l D_S^{p-l} D_3^l \nabla^k \Psi|_{L^p(S)} \leq \left(\frac{Q\rho_3}{\rho_1} \right)^l C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{1}{\rho_1^{p-l} \rho_3^l \rho_1^{k+1}} \quad (15.156)$$

and, therefore for $l = p$ we have

$$\begin{aligned} |\underline{u}^k |u|^p D_3^p \nabla^k \Psi|_{L^p(S)} &\leq \left(\frac{Q\rho_3}{\rho_1} \right)^p C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda)} \frac{1}{\rho_3^p \rho_1^{k+1}} \\ &\leq Q^p C_4 e^{(J-2)\delta} e^{(J-2)\Gamma(\lambda, \nu)} \frac{1}{\rho_1^p \rho_1^{k+1}}, \end{aligned} \quad (15.157)$$

15.7.1 The estimate of the other terms with commutators

To complete previous discussions we want to control the term, see equation 15.144

$$|\underline{u}^k |u|^l X D_S^{p-l} [\nabla, D_\lambda^{l-1}] \nabla^k \underline{\alpha}|_{L^p(S)}.$$

$$\begin{aligned} |\underline{u}^k |u|^l X D_S^{p-l} [\nabla, D_\lambda^{l-1}] \nabla^k \underline{\alpha}|_{L^p(S)} &\leq |X|_{\infty, S} |\underline{u}^k |u|^l D_S^{p-l} [\nabla, D_\lambda^{l-1}] \nabla^k \underline{\alpha}|_{L^p(S)} \\ &\leq |X|_{\infty, S} \sum_{s=0}^{l-2} |\underline{u}^k |u|^l D_S^{p-l} D_\lambda^{l-2-s} [\nabla, D_\lambda] D_\lambda^s \nabla^k \underline{\alpha}|_{L^p(S)} \end{aligned} \quad (15.158)$$

$D_\lambda^s \nabla^k \underline{\alpha}$ is a $(s+k+2)$ tensor, therefore we have

$$\begin{aligned} [\nabla_\mu, D_\lambda] D_\lambda^s \nabla^k \underline{\alpha}_{\tau_1 \dots \tau_{k+2}} &= \left[\sum_{j=1}^{s+k+2} \underline{C}_{\mu \lambda_j}^{\sigma_j} (D^s \nabla^k \underline{\alpha})_{\lambda_1 \dots \sigma_j \dots \lambda_s \tau_1 \dots \tau_{k+2}} + \Omega_{\underline{\lambda} \mu}^\rho \nabla_\rho (D^s \nabla^k \underline{\alpha})_{\lambda_1 \dots \lambda_s \tau_1 \dots \tau_{k+2}} \right] \\ \text{"} \leq \text{" } (s+k+2) \underline{C} (D^s \nabla^k \underline{\alpha}) + \Omega_{\underline{\lambda}} \cdot \nabla D_\lambda^s \nabla^k \underline{\alpha}. \end{aligned} \quad (15.159)$$

Therefore

$$\begin{aligned} |\underline{u}^k |u|^l X D_S^{p-l} [\nabla, D_\lambda^{l-1}] \nabla^k \underline{\alpha}|_{L^p(S)} &\leq |X|_{\infty, S} \sum_{s=0}^{l-2} (s+k+2) |\underline{u}^k |u|^l D_S^{p-l} D_\lambda^{l-2-s} \underline{C} D^s \nabla^k \underline{\alpha}|_{L^p(S)} \\ &+ \leq |X|_{\infty, S} \sum_{s=0}^{l-2} |\underline{u}^k |u|^l D_S^{p-l} D_\lambda^{l-2-s} \Omega_{\underline{\lambda}} \cdot \nabla D_\lambda^s \nabla^k \underline{\alpha}|_{L^p(S)} \\ &\leq |X|_{\infty, S} \sum_{s=0}^{l-2} (s+k+2) |\underline{u}^k |u|^l D_S^{p-l} D_\lambda^{l-2-s} \underline{C} D^s \nabla^k \underline{\alpha}|_{L^p(S)} \end{aligned}$$

$$\begin{aligned}
& + \leq |X|_{\infty, S} \sum_{s=0}^{l-2} |\underline{u}^k |u|^l D_S^{p-l} D_\lambda^{l-2-s} \Omega \underline{\chi} D_\lambda^s \nabla^{k+1} \underline{\alpha}|_{L^p(S)} + \left\{ l.o.t \right\} \\
& \leq \sigma |X|_{\infty, S} \sum_{s=0}^{l-2} (s+k+2) |\underline{u}^{k+1} |u|^{l-1} D_S^{p-l} D_\lambda^{l-2-s} \underline{C} D^s \nabla^k \underline{\alpha}|_{L^p(S)} \\
& + \leq \sigma |X|_{\infty, S} \sum_{s=0}^{l-2} |\underline{u}^{k+1} |u|^{l-1} D_S^{p-l} D_\lambda^{l-2-s} \Omega \underline{\chi} D_\lambda^s \nabla^{k+1} \underline{\alpha}|_{L^p(S)} + \left\{ l.o.t \right\} .
\end{aligned}$$

The first sum, treating \underline{C} as R is identical to the previous one and can be estimated in the same way, therefore we are left with estimating

$$\sigma |X|_{\infty, S} \sum_{s=0}^{l-2} |\underline{u}^{k+1} |u|^{l-1} D_S^{p-l} D_\lambda^{l-2-s} \Omega \underline{\chi} D_\lambda^s \nabla^{k+1} \underline{\alpha}|_{L^p(S)} \quad (15.160)$$

as the $\left\{ l.o.t \right\}$ is easy to control and has already been neglected. The second sum has the same structure as the first sum but simpler as it has $\Omega \underline{\chi}$ instead of \underline{C} and without the factor $(s+k+2)$ therefore it is easier to estimate.

We are left to estimate $\left\{ l.o.t.(I) \right\}$, see 15.143, this term can be easily estimated

following the same steps performed in calculation of $\left\{ l.o.t.(II) \right\}$

Corollary 4.4 follows choosing

$$\rho_2 \leq \rho_1 / Q. \quad (15.161)$$

Therefore in conclusion we have proved , omitting all the lower order terms,

$$\begin{aligned}
|\underline{u}^k |u|^p D_3^p \nabla^k \Psi|_{L^p(S)} & \leq H(\dots) \frac{1}{\rho_2^p \rho_1^{k+1}} \\
|\underline{u}^k |\underline{u}|^p D_4^p \nabla^k \Psi|_{L^p(S)} & \leq H(\dots) \frac{1}{\rho_2^p \rho_1^{k+1}} , \quad (15.162)
\end{aligned}$$

Remark:

In fact the second part of this result can be obtained proceeding as for the first part posing D_ν instead of D_λ and $|\underline{u}|$ instead of $|u|$.

16 Proof of Lemmas 3.3 and 3.6

The proof is a generalisation of the proof of theorem 3.4 and goes by induction, we assume these estimates hold for (J, P) and we prove they hold also for $(J+1, P-1)$, with $J+P=N$.

Let us start from the quantity we want to estimate:

$$\nabla^{p_0} \mathfrak{D}_\nu \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} U$$

where

$$U = \Omega^{-1} \text{tr} \chi \quad , \quad \sum_{s=0}^J p_s = P \quad , \quad J + P = N \quad .$$

We write the previous term in the following way, recalling 7.9,

$$\begin{aligned} & \nabla^{p_0} \mathfrak{D}_\nu \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} U \\ &= (\nabla^{p_0} \mathfrak{D}_\nu \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}+p_J})(\mathfrak{D}_\nu U) + (\nabla^{p_0} \mathfrak{D}_\nu \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}})([\nabla^{p_J}, \mathfrak{D}_\nu]U) \\ &= (\nabla^{p_0} \mathfrak{D}_\nu \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}+p_J}) \left(-\frac{\text{tr} \chi}{2} U - \Omega^{-1} |\hat{\chi}|^2 \right) + (\nabla^{p_0} \mathfrak{D}_\nu \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}})([\mathfrak{D}_\nu, \nabla^{p_J}]U) \quad . \end{aligned} \tag{16.1}$$

We use this last expression to prove inductively the desired estimate Recall that, see Appendix 14,

$$[\nabla_\mu, \mathfrak{D}_\nu] U_{\nu_1 \dots \nu_k} = - \sum_{j=1}^k C_{\mu \nu_j}^{\sigma_j} U_{\nu_1 \dots \sigma_j \dots \nu_k} + \chi_\mu^\rho (\nabla_\rho U)_{\nu_1 \dots \nu_k} \tag{16.2}$$

where

$$C_{\mu \nu_j}^{\sigma_j} = \Omega \left[(\chi_{\mu \nu_j} \underline{\eta}^{\sigma_j} - \chi_\mu^{\sigma_j} \underline{\eta}_{\nu_j}) + \Omega \theta_\mu^C \theta_{\nu_j}^D R^{\sigma_j}(\cdot, e_C, e_4, e_D) \right] \quad . \tag{16.3}$$

Therefore

$$\begin{aligned} [\nabla^{p_J}, \mathfrak{D}_\nu] U &= \sum_{q=0}^{p_J-1} \nabla^q [\nabla, \mathfrak{D}_\nu] \nabla^{p_J-q-1} U = \sum_{q=0}^{p_J-1} \nabla^q \left(-(p_J - q - 1) C \nabla^{p_J-q-1} U + \Omega \chi \nabla^{p_J-q} U \right) \\ &= - \sum_{q=0}^{p_J-1} (p_J - q - 1) \sum_{h=0}^q \binom{q}{h} (\nabla^h C) \nabla^{p_J-1-h} U + \sum_{q=0}^{p_J-1} \sum_{j=0}^q \binom{q}{j} (\nabla^j \chi) \nabla^{p_J-h} U \quad . \end{aligned} \tag{16.4}$$

Using the relation,

$$[\nabla^{p_J}, \mathfrak{D}_\nu] U = - \sum_{q=0}^{p_J-1} (p_J - q - 1) \sum_{h=0}^q \binom{q}{h} (\nabla^h C) \nabla^{p_J-1-h} U + \sum_{q=0}^{p_J-1} \sum_{h=0}^q \binom{q}{h} (\nabla^h \chi) \nabla^{p_J-h} U$$

we estimate the first term of the r.h.s. of equation 16.1 to check the inductive assumptions; defining

$$\sum_{s=0}^J p_s = P \quad , \quad \sum_{s=1}^{J-1} l_s = L \quad , \quad \sum_{s=0}^{J-1} h_s = H \quad ,$$

we have

$$\begin{aligned}
& |r^{1+J+P+2\sigma(P)-\frac{2}{p}} \nabla^{p_0} \mathbf{D}_\nu \nabla^{p_1} \mathbf{D}_\nu \nabla^{p_2} \dots \mathbf{D}_\nu \nabla^{p_{J-1}+p_J} (\text{tr} \chi U) |_{p,S} \\
& \leq \sum_{l_1=0}^1 \dots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \dots \sum_{h_{J-2}=0}^{p_{J-2}} \sum_{h_{J-1}=0}^{p_{J-1}+p_J} \binom{p_0}{h_0} \binom{p_1}{h_1} \dots \binom{p_{J-2}}{h_{J-2}} \binom{p_{J-1}+p_J}{h_{J-1}} \\
& |r^{1+J+P+2\sigma(P)-\frac{2}{p}} (\nabla^{h_0} \mathbf{D}_\nu^{l_1} \nabla^{h_1} \dots \mathbf{D}_\nu^{l_{J-2}} \nabla^{h_{J-2}} \mathbf{D}_\nu^{l_{J-1}} \nabla^{h_{J-1}} \text{tr} \chi) \cdot \\
& \cdot (\nabla^{p_0-h_0} \mathbf{D}_\nu^{1-l_1} \nabla^{p_1-h_1} \dots \mathbf{D}_\nu^{1-l_{J-2}} \nabla^{p_{J-2}-h_{J-2}} \mathbf{D}_\nu^{1-l_{J-1}} \nabla^{p_{J-1}+p_J-h_{J-1}} U) |_{p,S} \\
& \leq \sum_{l_1=0}^1 \dots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \dots \sum_{h_{J-2}=0}^{p_{J-2}} \sum_{h_{J-1}=0}^{p_{J-1}+p_J} \chi \left(L+H \leq \left\lfloor \frac{J+P}{2} \right\rfloor \right) \binom{p_0}{h_0} \binom{p_1}{h_1} \dots \binom{p_{J-2}}{h_{J-2}} \binom{p_{J-1}+p_J}{h_{J-1}} \\
& \frac{1}{r^{1+2\sigma(H)+2\sigma(P-H)-2\sigma(P)}} |r^{1+L+H+2\sigma(H)} (\nabla^{h_0} \mathbf{D}_\nu^{l_1} \nabla^{h_1} \dots \mathbf{D}_\nu^{l_{J-2}} \nabla^{h_{J-2}} \mathbf{D}_\nu^{l_{J-1}} \nabla^{h_{J-1}} \text{tr} \chi) |_{\infty,S} \\
& |r^{1+J+P-(L+H)+2\sigma(P-H)-\frac{2}{p}} (\nabla^{p_0-h_0} \mathbf{D}_\nu^{1-l_1} \nabla^{p_1-h_1} \dots \mathbf{D}_\nu^{1-l_{J-2}} \nabla^{p_{J-2}-h_{J-2}} \mathbf{D}_\nu^{1-l_{J-1}} \nabla^{p_{J-1}+p_J-h_{J-1}} U) |_{p,S} \\
& + \sum_{l_1=0}^1 \dots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \dots \sum_{h_{J-2}=0}^{p_{J-2}} \sum_{h_{J-1}=0}^{p_{J-1}+p_J} \chi \left(L+H \geq \left\lfloor \frac{J+P}{2} \right\rfloor + 1 \right) \binom{p_0}{h_0} \binom{p_1}{h_1} \dots \binom{p_{J-2}}{h_{J-2}} \binom{p_{J-1}+p_J}{h_{J-1}} \\
& \frac{1}{r^{1+2\sigma(H)+2\sigma(P-H)-2\sigma(P)}} |r^{1+L+H+2\sigma(H)-\frac{2}{p}} (\nabla^{h_0} \mathbf{D}_\nu^{l_1} \nabla^{h_1} \dots \mathbf{D}_\nu^{l_{J-2}} \nabla^{h_{J-2}} \mathbf{D}_\nu^{l_{J-1}} \nabla^{h_{J-1}} \text{tr} \chi) |_{p,S} \\
& |r^{1+J+P-(L+H)+2\sigma(P-H)} (\nabla^{p_0-h_0} \mathbf{D}_\nu^{1-l_1} \nabla^{p_1-h_1} \dots \mathbf{D}_\nu^{1-l_{J-2}} \nabla^{p_{J-2}-h_{J-2}} \mathbf{D}_\nu^{1-l_{J-1}} \nabla^{p_{J-1}+p_J-h_{J-1}} U) |_{\infty,S} .
\end{aligned} \tag{16.5}$$

Let us consider the first sum, $[(I)]$, and, denoting $K = H + L$, we have, using as inductive assumptions the estimates of Lemma 3.5, with $J - 1$ instead of J ,

$$\begin{aligned}
[(I)] & \leq \sum_{k=0}^{J+P} \left[\sum_{l_1=0}^1 \dots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \dots \sum_{h_{J-2}=0}^{p_{J-2}} \sum_{h_{J-1}=0}^{p_{J-1}+p_J} \binom{p_0}{h_0} \binom{p_1}{h_1} \dots \binom{p_{J-2}}{h_{J-2}} \binom{p_{J-1}+p_J}{h_{J-1}} \delta(H+L-k) \right] \\
& \chi \left(L+H \leq \left\lfloor \frac{J+P}{2} \right\rfloor \right) \left[\frac{1}{r^{1+\sigma(H)+\sigma(P-H)-\sigma(P)}} \frac{(L+H+1)!}{(L+H+1)^\alpha} \frac{e^{(L+H-1)(\delta+\Gamma)}}{\rho_2^{L+H+1}} \right] \left[\frac{(J+P-1-(L+H))!}{(J+P-1-(L+H))^\alpha} \frac{e^{(J+P-(L+H)-3)(\delta+\Gamma)}}{\rho_2^{(J+P-(L+H)-1)}} \right]
\end{aligned} \tag{16.6}$$

Observe that, denoting with ∇ a generic derivative operator we have, recalling that $\sum_{s=0}^J p_s = P$, $\sum_{s=1}^{J-1} l_s = L$, $\sum_{s=0}^{J-1} h_s = H$,

$$\begin{aligned}
\nabla^{(J-1)+P} AB & = \sum_{k=0}^{J+P-1} \binom{J+P-1}{k} \nabla^k A \nabla^{J+P-1-k} B = \nabla^{p_0} \nabla \nabla^{p_1} \nabla \dots \nabla^{p_{j-1}+p_j} AB \\
& = \sum_{l_1=0}^1 \dots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \dots \sum_{h_{J-2}=0}^{p_{J-2}} \sum_{h_{J-1}=0}^{p_{J-1}+p_J} \binom{p_0}{h_0} \binom{p_1}{h_1} \dots \binom{p_{J-2}}{h_{J-2}} \binom{p_{J-1}+p_J}{h_{J-1}} \\
& \cdot \left(\nabla^{h_0} \nabla^{l_1} \nabla^{h_1} \dots \nabla^{l_{J-2}} \nabla^{h_{J-2}} \nabla^{l_{J-1}} \nabla^{h_{J-1}} A \right) \left(\nabla^{p_0-h_0} \nabla^{1-l_1} \nabla^{p_1-h_1} \dots \nabla^{1-l_{J-2}} \nabla^{p_{J-2}-h_{J-2}} \nabla^{1-l_{J-1}} \nabla^{p_{J-1}+p_J-h_{J-1}} B \right) \\
& = \sum_{k=0}^{J+P-1} \left[\sum_{l_1=0}^1 \dots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \dots \sum_{h_{J-2}=0}^{p_{J-2}} \sum_{h_{J-1}=0}^{p_{J-1}+p_J} \binom{p_0}{h_0} \binom{p_1}{h_1} \dots \binom{p_{J-2}}{h_{J-2}} \binom{p_{J-1}+p_J}{h_{J-1}} \delta(H+L-k) \right]
\end{aligned}$$

$$\cdot \nabla^{H+L} A \nabla^{J+P-1-(H+L)} B \quad (16.7)$$

$$= \sum_{k=0}^{J+P-1} \left[\sum_{l_1=0}^1 \cdots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \cdots \sum_{h_{J-2}=0}^{p_{J-2}} \sum_{h_{J-1}=0}^{p_{J-1}+p_J} \binom{p_0}{h_0} \binom{p_1}{h_1} \cdots \binom{p_{J-2}}{h_{J-2}} \binom{p_{J-1}+p_J}{h_{J-1}} \delta(H+L-k) \right] \cdot \nabla^k A \nabla^{J+P-1-k} B, \quad (16.8)$$

which implies

$$\left[\sum_{l_1=0}^1 \cdots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \cdots \sum_{h_{J-2}=0}^{p_{J-2}} \sum_{h_{J-1}=0}^{p_{J-1}+p_J} \binom{p_0}{h_0} \binom{p_1}{h_1} \cdots \binom{p_{J-2}}{h_{J-2}} \binom{p_{J-1}+p_J}{h_{J-1}} \delta(H+L-k) \right] = \binom{J+P-1}{k}. \quad (16.9)$$

Therefore

$$\begin{aligned} [(I)] &\leq \sum_{k=0}^{\lfloor \frac{J+P-1}{2} \rfloor} \binom{J+P-1}{k} \frac{\hat{F}}{r} \left[\frac{(k+1)!}{(k+1)^\alpha} \frac{e^{(k-1)(\delta+\Gamma)}}{\rho_2^{k+1}} \right] \left[\frac{(J+P-1-k)!}{(J+P-1-k)^\alpha} \frac{e^{(J+P-k-3)(\delta+\Gamma)}}{\rho_2^{(J+P-k-1)}} \right] \\ &+ \sum_{k=\lfloor \frac{J+P-1}{2} \rfloor+1}^{J+P-1} \binom{J+P-1}{k} \frac{\hat{F}}{r} \left[\frac{(k+1)!}{(k+1)^\alpha} \frac{e^{(k-1)(\delta+\Gamma)}}{\rho_2^{k+1}} \right] \left[\frac{(J+P-1-k)!}{(J+P-1-k)^\alpha} \frac{e^{(J+P-k-3)(\delta+\Gamma)}}{\rho_2^{(J+P-k-1)}} \right]. \end{aligned} \quad (16.10)$$

(16.11)

Let us estimate the first sum:

$$\leq \left(\hat{F} \frac{(J+P)!}{(J+P)^\alpha} \frac{e^{(J+P-2)(\delta+\Gamma)}}{\rho_2^{J+P}} \right) \left(\frac{e^{-2(\delta+\Gamma)}}{r} \right) \left[\frac{(J+P)^\alpha}{(J+P)!} \sum_{k=0}^{\lfloor \frac{J+P-1}{2} \rfloor} \frac{(J+P-1)!}{(J+P-1-k)!k!} \frac{(k+1)!}{(k+1)^\alpha} \frac{(J+P-1-k)!}{(J+P-1-k)^\alpha} \right]$$

and

$$\begin{aligned} &\left[\frac{(J+P)^\alpha}{(J+P)!} \sum_{k=0}^{\lfloor \frac{J+P-1}{2} \rfloor} \frac{(J+P-1)!}{(J+P-1-k)!k!} \frac{(k+1)!}{(k+1)^\alpha} \frac{(J+P-1-k)!}{(J+P-1-k)^\alpha} \right] \quad (16.12) \\ &\leq c \left[\frac{(J+P-1)!(J+P)^\alpha}{(J+P)!(J+P-1-\lfloor \frac{J+P-1}{2} \rfloor)^\alpha} \sum_{k=0}^{\lfloor \frac{J+P-1}{2} \rfloor} \frac{1}{(k+1)^{\alpha-1}} \right] \leq C(\alpha) \end{aligned}$$

provided $\alpha > 2$. The second sum of the same term can be treated in the same way. We estimate in the same way the second term of equation 16.1. Let us consider now the most delicate term, namely,

$$(\nabla^{p_0} \mathbf{D}_\nu \nabla^{p_1} \mathbf{D}_\nu \nabla^{p_2} \cdots \mathbf{D}_\nu \nabla^{p_{J-1}}) ([\mathbf{D}_\nu, \nabla^{p_J}] U)$$

The estimate proceeds in the following way,

$$|[(II)]| \equiv \left| r^{1+J+P+2\sigma(P)-\frac{2}{p}} \nabla^{p_0} \mathbf{D}_\nu \nabla^{p_1} \mathbf{D}_\nu \nabla^{p_2} \cdots \mathbf{D}_\nu \nabla^{p_{J-1}} ([\mathbf{D}_\nu, \nabla^{p_J}] U) \right|_{p,S}$$

$$\begin{aligned}
&\leq \left| r^{1+J+P+2\sigma(P)-\frac{2}{p}} \nabla^{p_0} \mathfrak{D}_\nu \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \left(\sum_{h=0}^{p_J-1} \nabla^h [\mathfrak{D}_\nu, \nabla] \nabla^{p_J-1-h} U \right) \right|_{p,S} \\
&\leq \sum_{h=0}^{p_J-1} \left| r^{1+J+P+2\sigma(P)-\frac{2}{p}} \nabla^{p_0} \mathfrak{D}_\nu \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}+h} \left([\mathfrak{D}_\nu, \nabla] \nabla^{p_J-1-h} U \right) \right|_{p,S}
\end{aligned}$$

As the following holds

$$[\mathfrak{D}_\nu, \nabla] \nabla^{p_J-1-h} U = \left(-(p_J - 1 - h) C \nabla^{p_J-1-h} U + \Omega \chi \nabla^{p_J-h} U \right), \quad (16.13)$$

$$\begin{aligned}
&|[(II)]| \leq \\
&\leq \sum_{h=0}^{p_J-1} \left| r^{1+J+P+2\sigma(P)-\frac{2}{p}} \nabla^{p_0} \mathfrak{D}_\nu \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}+h} \left(-(p_J - 1 - h) C \nabla^{p_J-1-h} U + \Omega \chi \nabla^{p_J-h} U \right) \right|_{p,S} \\
&\leq \sum_{h=0}^{p_J-1} (p_J - 1 - h) \left| r^{1+J+P+2\sigma(P)-\frac{2}{p}} \nabla^{p_0} \mathfrak{D}_\nu \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}+h} (C \nabla^{p_J-1-h} U) \right|_{p,S} \\
&+ \sum_{h=0}^{p_J-1} \left| r^{1+J+P+2\sigma(P)-\frac{2}{p}} \nabla^{p_0} \mathfrak{D}_\nu \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}+h} (\Omega \chi \nabla^{p_J-h-1} \nabla U) \right|_{p,S} \\
&\leq \sum_{h=0}^{p_J-1} (p_J - 1 - h) \sum_{l_1=0}^1 \dots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \dots \sum_{h_{J-2}=0}^{p_{J-2}} \sum_{h_{J-1}=0}^{p_{J-1}+h} \binom{p_0}{h_0} \binom{p_1}{h_1} \dots \binom{p_{J-2}}{h_{J-2}} \binom{p_{J-1}+h}{h_{J-1}} \\
&\left| r^{1+J+P+2\sigma(P)-\frac{2}{p}} (\nabla^{h_0} \mathfrak{D}_\nu^{l_1} \nabla^{h_1} \dots \mathfrak{D}_\nu^{l_{J-2}} \nabla^{h_{J-2}} \mathfrak{D}_\nu^{l_{J-1}} \nabla^{h_{J-1}} C) (\nabla^{p_0-h_0} \mathfrak{D}_\nu^{1-l_1} \nabla^{p_1-h_1} \dots \mathfrak{D}_\nu^{1-l_{J-2}} \nabla^{p_{J-2}-h_{J-2}} \mathfrak{D}_\nu^{1-l_{J-1}} \nabla^{p_{J-1}+p_J-h_{J-1}-1} U) \right|_{p,S}
\end{aligned}$$

$$+ \sum_{h=0}^{p_J-1} \sum_{l_1=0}^1 \dots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \dots \sum_{h_{J-2}=0}^{p_{J-2}} \sum_{h_{J-1}=0}^{p_{J-1}+h} \binom{p_0}{h_0} \binom{p_1}{h_1} \dots \binom{p_{J-2}}{h_{J-2}} \binom{p_{J-1}+h}{h_{J-1}}$$

$$\left| r^{1+J+P+2\sigma(P)-\frac{2}{p}} (\nabla^{h_0} \mathfrak{D}_\nu^{l_1} \nabla^{h_1} \dots \mathfrak{D}_\nu^{l_{J-2}} \nabla^{h_{J-2}} \mathfrak{D}_\nu^{l_{J-1}} \nabla^{h_{J-1}} \Omega \chi) (\nabla^{p_0-h_0} \mathfrak{D}_\nu^{1-l_1} \nabla^{p_1-h_1} \dots \mathfrak{D}_\nu^{1-l_{J-2}} \nabla^{p_{J-2}-h_{J-2}} \mathfrak{D}_\nu^{1-l_{J-1}} \nabla^{p_{J-1}+p_J-h_{J-1}-1} \nabla U) \right|_{p,S}$$

Let us consider the part of the sum with $L + H \leq \left\lfloor \frac{J+P}{2} \right\rfloor$,

$$|[(II)]| \leq \quad (16.14)$$

$$\begin{aligned}
&\sum_{l_1=0}^1 \dots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \dots \sum_{h_{J-2}=0}^{p_{J-2}} \binom{p_0}{h_0} \binom{p_1}{h_1} \dots \binom{p_{J-2}}{h_{J-2}} \left(\sum_{h=0}^{p_J-1} (p_J - 1 - h) \sum_{h_{J-1}=0}^{p_{J-1}+h} \binom{p_{J-1}+h}{h_{J-1}} \right) \chi \left(L + H \leq \left\lfloor \frac{J+P}{2} \right\rfloor \right) \\
&\left| r^{1+L+H} (\nabla^{h_0} \mathfrak{D}_\nu^{l_1} \nabla^{h_1} \dots \mathfrak{D}_\nu^{l_{J-2}} \nabla^{h_{J-2}} \mathfrak{D}_\nu^{l_{J-1}} \nabla^{h_{J-1}} C) \right|_{\infty,S} \cdot \\
&\cdot \left| r^{J+P-(L+H)+2\sigma(P)-\frac{2}{p}} (\nabla^{p_0-h_0} \mathfrak{D}_\nu^{1-l_1} \nabla^{p_1-h_1} \dots \mathfrak{D}_\nu^{1-l_{J-2}} \nabla^{p_{J-2}-h_{J-2}} \mathfrak{D}_\nu^{1-l_{J-1}} \nabla^{p_{J-1}+p_J-h_{J-1}-1} U) \right|_{p,S} \\
&+ \sum_{h=0}^{p_J-1} \sum_{l_1=0}^1 \dots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \dots \sum_{h_{J-2}=0}^{p_{J-2}} \sum_{h_{J-1}=0}^{p_{J-1}+h} \binom{p_0}{h_0} \binom{p_1}{h_1} \dots \binom{p_{J-2}}{h_{J-2}} \binom{p_{J-1}+h}{h_{J-1}} \chi \left(L + H \leq \left\lfloor \frac{J+P}{2} \right\rfloor \right)
\end{aligned}$$

$$\begin{aligned}
& \left| r^{1+L+H} (\nabla^{h_0} \mathfrak{D}_\nu^{l_1} \nabla^{h_1} \dots \mathfrak{D}_\nu^{l_{J-2}} \nabla^{h_{J-2}} \mathfrak{D}_\nu^{l_{J-1}} \nabla^{h_{J-1}} \Omega \chi) \right|_{\infty, S} \cdot \\
& \cdot \left| r^{J+P-(L+H)+2\sigma(P)-\frac{2}{p}} (\nabla^{p_0-h_0} \mathfrak{D}_\nu^{1-l_1} \nabla^{p_1-h_1} \dots \mathfrak{D}_\nu^{1-l_{J-2}} \nabla^{p_{J-2}-h_{J-2}} \mathfrak{D}_\nu^{1-l_{J-1}} \nabla^{p_{J-1}+p_J-h_{J-1}-1} \nabla U) \right|_{p, S} \\
& \leq \sum_{l_1=0}^1 \dots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \dots \sum_{h_{J-2}=0}^{p_{J-2}} \binom{p_0}{h_0} \binom{p_1}{h_1} \dots \binom{p_{J-2}}{h_{J-2}} \left(\sum_{h=0}^{p_J-1} (p_J-1-h) \sum_{h_{J-1}=0}^{p_{J-1}+h} \binom{p_{J-1}+h}{h_{J-1}} \right) \chi \left(L+H \leq \left\lfloor \frac{J+P}{2} \right\rfloor \right) \\
& \left| r^{1+L+H} (\nabla^{h_0} \mathfrak{D}_\nu^{l_1} \nabla^{h_1} \dots \mathfrak{D}_\nu^{l_{J-2}} \nabla^{h_{J-2}} \mathfrak{D}_\nu^{l_{J-1}} \nabla^{h_{J-1}} C) \right|_{\infty, S} \cdot \\
& \cdot \left| r^{J+P-(L+H)+2\sigma(P)-\frac{2}{p}} (\nabla^{p_0-h_0} \mathfrak{D}_\nu^{1-l_1} \nabla^{p_1-h_1} \dots \mathfrak{D}_\nu^{1-l_{J-2}} \nabla^{p_{J-2}-h_{J-2}} \mathfrak{D}_\nu^{1-l_{J-1}} \nabla^{p_{J-1}+p_J-h_{J-1}-1} U) \right|_{p, S} \\
& + \sum_{h=0}^{p_J-1} \sum_{l_1=0}^1 \dots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \dots \sum_{h_{J-2}=0}^{p_{J-2}} \sum_{h_{J-1}=0}^{p_{J-1}+h} \binom{p_0}{h_0} \binom{p_1}{h_1} \dots \binom{p_{J-2}}{h_{J-2}} \binom{p_{J-1}+h}{h_{J-1}} \chi \left(L+H \leq \left\lfloor \frac{J+P}{2} \right\rfloor \right) \\
& \left| r^{1+L+H} (\nabla^{h_0} \mathfrak{D}_\nu^{l_1} \nabla^{h_1} \dots \mathfrak{D}_\nu^{l_{J-2}} \nabla^{h_{J-2}} \mathfrak{D}_\nu^{l_{J-1}} \nabla^{h_{J-1}} \Omega \chi) \right|_{\infty, S} \cdot \\
& \cdot \left| r^{J+P-(L+H)+2\sigma(P)-\frac{2}{p}} (\nabla^{p_0-h_0} \mathfrak{D}_\nu^{1-l_1} \nabla^{p_1-h_1} \dots \mathfrak{D}_\nu^{1-l_{J-2}} \nabla^{p_{J-2}-h_{J-2}} \mathfrak{D}_\nu^{1-l_{J-1}} \nabla^{p_{J-1}+p_J-h_{J-1}-1} \nabla U) \right|_{p, S} \\
& \leq \sum_{l_1=0}^1 \dots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \dots \sum_{h_{J-2}=0}^{p_{J-2}} \binom{p_0}{h_0} \binom{p_1}{h_1} \dots \binom{p_{J-2}}{h_{J-2}} \left(\sum_{h=0}^{p_J-1} (p_J-1-h) \sum_{h_{J-1}=0}^{p_{J-1}+h} \binom{p_{J-1}+h}{h_{J-1}} \right) \chi \left(L+H \leq \left\lfloor \frac{J+P}{2} \right\rfloor \right) \\
& \frac{1}{r^{2\sigma(P-H-1)}} \left| r^{3+L+H} (\nabla^{h_0} \mathfrak{D}_\nu^{l_1} \nabla^{h_1} \dots \mathfrak{D}_\nu^{l_{J-2}} \nabla^{h_{J-2}} \mathfrak{D}_\nu^{l_{J-1}} \nabla^{h_{J-1}} C) \right|_{\infty, S} \cdot \\
& \cdot \left| r^{J+P-(L+H)+2\sigma(P-H-1)-\frac{2}{p}} (\nabla^{p_0-h_0} \mathfrak{D}_\nu^{1-l_1} \nabla^{p_1-h_1} \dots \mathfrak{D}_\nu^{1-l_{J-2}} \nabla^{p_{J-2}-h_{J-2}} \mathfrak{D}_\nu^{1-l_{J-1}} \nabla^{p_{J-1}+p_J-h_{J-1}-1} U) \right|_{p, S} \\
& + \sum_{h=0}^{p_J-1} \sum_{l_1=0}^1 \dots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \dots \sum_{h_{J-2}=0}^{p_{J-2}} \sum_{h_{J-1}=0}^{p_{J-1}+h} \binom{p_0}{h_0} \binom{p_1}{h_1} \dots \binom{p_{J-2}}{h_{J-2}} \binom{p_{J-1}+h}{h_{J-1}} \chi \left(L+H \leq \left\lfloor \frac{J+P}{2} \right\rfloor \right) \\
& \left| r^{1+L+H+\sigma(H)} (\nabla^{h_0} \mathfrak{D}_\nu^{l_1} \nabla^{h_1} \dots \mathfrak{D}_\nu^{l_{J-2}} \nabla^{h_{J-2}} \mathfrak{D}_\nu^{l_{J-1}} \nabla^{h_{J-1}} \Omega \chi) \right|_{\infty, S} \cdot \\
& \cdot \frac{1}{r^{1+\sigma(H)}} \left| r^{3+(J-1)+P-(L+H)-\frac{2}{p}} (\nabla^{p_0-h_0} \mathfrak{D}_\nu^{1-l_1} \nabla^{p_1-h_1} \dots \mathfrak{D}_\nu^{1-l_{J-2}} \nabla^{p_{J-2}-h_{J-2}} \mathfrak{D}_\nu^{1-l_{J-1}} \nabla^{p_{J-1}+p_J-h_{J-1}-1} \nabla U) \right|_{p, S}
\end{aligned}$$

Remark:

Observe that in the term product of the two previous factors there is one derivative \mathfrak{D}_ν less and the same number of ∇ derivatives, therefore the binomial coefficient of the second factor is $((J-1)+P-(L+H)+1)! = (J+P-(L+H))!$. Viceversa in the previous term of the sum where C appears there is one derivative \mathfrak{D}_ν less and also one derivative ∇ less.

Therefore

$$\begin{aligned}
& |[(II)]| \leq \\
& \leq \sum_{l_1=0}^1 \dots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \dots \sum_{h_{J-2}=0}^{p_{J-2}} \binom{p_0}{h_0} \binom{p_1}{h_1} \dots \binom{p_{J-2}}{h_{J-2}} \left(\sum_{h=0}^{p_J-1} (p_J-1-h) \sum_{h_{J-1}=0}^{p_{J-1}+h} \binom{p_{J-1}+h}{h_{J-1}} \right) \chi \left(L+H \leq \left\lfloor \frac{J+P}{2} \right\rfloor \right)
\end{aligned} \tag{16.15}$$

$$\begin{aligned}
& \frac{1}{r^{2\sigma(P-H-1)}} \left[\frac{(L+H+2)!}{(L+H+2)^\alpha} \frac{e^{(L+H)(\delta+\Gamma(\lambda))}}{\rho_2^{L+H+2}} \frac{(J+P-(L+H)-2)!}{(J+P-(L+H)-2)^\alpha} \frac{e^{(J+P-(L+H)-4)(\delta+\Gamma(\lambda))}}{\rho_2^{J+P-(L+H)-2}} \right] \\
& + \sum_{h=0}^{p_{J-1}-1} \sum_{l_1=0}^1 \cdots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \cdots \sum_{h_{J-2}=0}^{p_{J-2}} \sum_{h_{J-1}=0}^{p_{J-1}+h} \binom{p_0}{h_0} \binom{p_1}{h_1} \cdots \binom{p_{J-2}}{h_{J-2}} \binom{p_{J-1}+h}{h_{J-1}} \chi \left(L+H \leq \left\lfloor \frac{J+P}{2} \right\rfloor \right) \\
& \frac{1}{r^{1+\sigma(H)}} \left[\frac{(L+H+1)!}{(L+H+1)^\alpha} \frac{e^{((L+H)-1)(\delta+\Gamma(\lambda))}}{\rho_2^{L+H+1}} \frac{(J+P-1-(L+H))!}{(J+P-1-(L+H))^\alpha} \frac{e^{(J+P-(L+H)-3)(\delta+\Gamma(\lambda))}}{\rho_2^{J+P-(L+H)-1}} \right]
\end{aligned}$$

which we rewrite, denoting $k = L + H$,

$$\begin{aligned}
& |[II]| \leq \tag{16.16} \\
& \leq \sum_{k=0}^{\left\lfloor \frac{J+P}{2} \right\rfloor} \delta(L+H-k) \left\{ \sum_{l_1=0}^1 \cdots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \cdots \sum_{h_{J-2}=0}^{p_{J-2}} \binom{p_0}{h_0} \binom{p_1}{h_1} \cdots \binom{p_{J-2}}{h_{J-2}} \left(\sum_{h=0}^{p_{J-1}-1} (p_{J-1}-1-h) \sum_{h_{J-1}=0}^{p_{J-1}+h} \binom{p_{J-1}+h}{h_{J-1}} \right) \right\} \\
& \frac{1}{r^{2\sigma(P-H-1)}} \left[\frac{(L+H+2)!}{(L+H+2)^\alpha} \frac{e^{(L+H)(\delta+\Gamma(\lambda))}}{\rho_2^{L+H+2}} \frac{(J+P-(L+H)-2)!}{(J+P-(L+H)-2)^\alpha} \frac{e^{(J+P-(L+H)-4)(\delta+\Gamma(\lambda))}}{\rho_2^{J+P-(L+H)-2}} \right] \\
& + \sum_{k=0}^{\left\lfloor \frac{J+P}{2} \right\rfloor} \delta(L+H-k) \left\{ \sum_{h=0}^{p_{J-1}-1} \sum_{l_1=0}^1 \cdots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \cdots \sum_{h_{J-2}=0}^{p_{J-2}} \sum_{h_{J-1}=0}^{p_{J-1}+h} \binom{p_0}{h_0} \binom{p_1}{h_1} \cdots \binom{p_{J-2}}{h_{J-2}} \binom{p_{J-1}+h}{h_{J-1}} \right\} \\
& \frac{1}{r^{1+\sigma(H)}} \left[\frac{(L+H+1)!}{(L+H+1)^\alpha} \frac{e^{((L+H)-1)(\delta+\Gamma(\lambda))}}{\rho_2^{L+H+1}} \frac{(J+P-1-(L+H))!}{(J+P-1-(L+H))^\alpha} \frac{e^{(J+P-(L+H)-3)(\delta+\Gamma(\lambda))}}{\rho_2^{J+P-(L+H)-1}} \right] \tag{16.17}
\end{aligned}$$

The following holds, see before eqs. (15.11),

$$\sum_{k=J}^{N-2} (N-k-1) \binom{k}{J} = \binom{N}{N-2-J} \tag{16.18}$$

Observe that, denoting

$$h_{J-1} \equiv J \quad , \quad p_{J-1} + h \equiv k \quad , \quad p_{J-1} + p_J \equiv N$$

$$\begin{aligned}
& \sum_{h=0}^{p_{J-1}-1} (p_{J-1}-1-h) \sum_{h_{J-1}=0}^{p_{J-1}+h} \binom{p_{J-1}+h}{h_{J-1}} = \sum_{k=p_{J-1}}^{N-1} (N-k-1) \sum_{J=0}^k \binom{k}{J} = \sum_{J=0}^{N-1} \sum_{k=J}^{N-1} (N-k-1) \binom{k}{J} \\
& = \sum_{J=0}^{N-1} \sum_{k=J}^{N-2} (N-k-1) \binom{k}{J} = \sum_{J=0}^{N-2} \sum_{k=J}^{N-2} (N-k-1) \binom{k}{J} = \sum_{J=0}^{N-2} \binom{N}{N-2-J} \\
& = \sum_{J=0}^{N-2} \binom{N}{N-2-J} = \sum_{h_{J-1}=0}^{p_{J-1}+p_J-2} \binom{p_{J-1}+p_J}{h_{J-1}+2} = \sum_{\hat{h}_{J-1}=2}^{p_{J-1}+p_J} \binom{p_{J-1}+p_J}{\hat{h}_{J-1}} \\
& \leq \sum_{\hat{h}_{J-1}=0}^{p_{J-1}+p_J} \binom{p_{J-1}+p_J}{\hat{h}_{J-1}} \quad , \tag{16.19}
\end{aligned}$$

The following holds, see before

$$\sum_{H=J}^{N-1} \binom{H}{k} = \binom{N}{k+1} = \binom{N}{N-k-1} \quad (16.20)$$

Therefore denoting $J + P - 1 - (p_J - h) = H$

$$\sum_{h=0}^{p_J-1} \binom{J+P-1-(p_J-h)}{k} = \sum_{H=J+P-1-p_J}^{J+P-2} \binom{H}{k} \leq \sum_{H=k}^{J+P-2} \binom{H}{k} = \binom{J+P-1}{k+1}, \quad (16.21)$$

where the inequality has a meaning if we sum only non negative terms, which is the present case; therefore the previous estimate can be rewritten as

$$\begin{aligned} & |[(II)]| \leq \quad (16.22) \\ & \leq \sum_{k=0}^{\lfloor \frac{J+P}{2} \rfloor} \delta(L+H-k) \left\{ \sum_{l_1=0}^1 \cdots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \cdots \sum_{h_{J-2}=0}^{p_{J-2}} \sum_{\hat{h}_{J-1}=2}^{p_{J-1}+p_J} \binom{p_0}{h_0} \binom{p_1}{h_1} \cdots \binom{p_{J-2}}{h_{J-2}} \binom{p_{J-1}+p_J}{\hat{h}_{J-1}} \right\} \\ & \quad \frac{1}{r^{2\sigma(P-H-1)}} \left[\frac{(L+H+2)!}{(L+H+2)^\alpha} \frac{e^{((L+H))(\delta+\Gamma(\lambda))}}{\rho_2^{L+H+2}} \frac{(J+P-(L+H)-2)!}{(J+P-(L+H)-2)^\alpha} \frac{e^{(J+P-(L+H)-4)(\delta+\Gamma(\lambda))}}{\rho_2^{J+P-(L+H)-2}} \right] \\ & + \sum_{k=0}^{\lfloor \frac{J+P}{2} \rfloor} \delta(L+H-k) \left\{ \sum_{h=0}^{p_J-1} \sum_{l_1=0}^1 \cdots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \cdots \sum_{h_{J-2}=0}^{p_{J-2}} \sum_{h_{J-1}=0}^{p_{J-1}+h} \binom{p_0}{h_0} \binom{p_1}{h_1} \cdots \binom{p_{J-2}}{h_{J-2}} \binom{p_{J-1}+h}{h_{J-1}} \right\} \\ & \quad \frac{1}{r^{1+\sigma(H)}} \left[\frac{(L+H+1)!}{(L+H+1)^\alpha} \frac{e^{((L+H)-1)(\delta+\Gamma(\lambda))}}{\rho_2^{L+H+1}} \frac{(J+P-1-(L+H))!}{(J+P-1-(L+H))^\alpha} \frac{e^{(J+P-(L+H)-3)(\delta+\Gamma(\lambda))}}{\rho_2^{J+P-(L+H)-1}} \right]. \end{aligned}$$

Recall that, see 16.9,

$$\begin{aligned} & \delta(L+H-k) \left\{ \sum_{l_1=0}^1 \cdots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \cdots \sum_{h_{J-2}=0}^{p_{J-2}} \sum_{\hat{h}_{J-1}=2}^{p_{J-1}+p_J} \binom{p_0}{h_0} \binom{p_1}{h_1} \cdots \binom{p_{J-2}}{h_{J-2}} \binom{p_{J-1}+p_J}{\hat{h}_{J-1}} \right\} \\ & \text{“} \leq \text{” } (L+H-k) \left\{ \sum_{l_1=0}^1 \cdots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \cdots \sum_{h_{J-2}=0}^{p_{J-2}} \sum_{\hat{h}_{J-1}=0}^{p_{J-1}+p_J} \binom{p_0}{h_0} \binom{p_1}{h_1} \cdots \binom{p_{J-2}}{h_{J-2}} \binom{p_{J-1}+p_J}{\hat{h}_{J-1}} \right\} = \binom{J+P-1}{k} \end{aligned}$$

and

$$\begin{aligned} & \delta(L+H-k) \left\{ \sum_{h=0}^{p_J-1} \sum_{l_1=0}^1 \cdots \sum_{l_{J-2}=0}^1 \sum_{l_{J-1}=0}^1 \sum_{h_0=0}^{p_0} \sum_{h_1=0}^{p_1} \cdots \sum_{h_{J-2}=0}^{p_{J-2}} \sum_{h_{J-1}=0}^{p_{J-1}+h} \binom{p_0}{h_0} \binom{p_1}{h_1} \cdots \binom{p_{J-2}}{h_{J-2}} \binom{p_{J-1}+h}{h_{J-1}} \right\} \\ & = \sum_{h=0}^{p_J-1} \binom{J+P-1-(p_J-h)}{k} \quad (16.23) \end{aligned}$$

and substituting in the previous expression, we obtain

$$|[(II)]| \leq \sum_{k=0}^{\lfloor \frac{J+P}{2} \rfloor} \binom{J+P-1}{k} \left[\frac{(k+2)!}{(k+2)^\alpha} \frac{e^{k(\delta+\Gamma(\lambda))}}{\rho_2^{k+2}} \frac{(J+P-k-2)!}{(J+P-k-2)^\alpha} \frac{e^{(J+P-k-4)(\delta+\Gamma(\lambda))}}{\rho_2^{J+P-k-2}} \right] \\ + \frac{1}{r} \sum_{k=0}^{\lfloor \frac{J+P}{2} \rfloor} \sum_{h=0}^{p_J-1} \chi \left(k \leq (J+P+h-1-p_J) \right) \binom{J+P-1-(p_J-h)}{k} \left[\frac{(k+1)!}{(k+1)^\alpha} \frac{e^{(k-1)(\delta+\Gamma(\lambda))}}{\rho_2^{k+1}} \frac{(J+P-k-1)!}{(J+P-k-1)^\alpha} \frac{e^{(J+P-k-3)(\delta+\Gamma(\lambda))}}{\rho_2^{J+P-k-1}} \right]$$

Let us estimate the two sums separately

$$\sum_{k=0}^{\lfloor \frac{J+P}{2} \rfloor} \binom{J+P-1}{k} \left[\frac{(k+2)!}{(k+2)^\alpha} \frac{e^{k(\delta+\Gamma(\lambda))}}{\rho_2^{k+2}} \frac{(J+P-k-2)!}{(J+P-k-2)^\alpha} \frac{e^{(J+P-k-4)(\delta+\Gamma(\lambda))}}{\rho_2^{J+P-k-2}} \right] \\ \leq c \left(\frac{(J+P)!}{(J+P)^\alpha} \frac{e^{(J+P-2)(\delta+\Gamma(\lambda))}}{\rho_2^{J+P}} \right) \left(e^{-2(\delta+\Gamma(\lambda))} \right) \cdot \left\{ \frac{(J+P)^\alpha}{(J+P)!} \sum_{k=0}^{\lfloor \frac{J+P}{2} \rfloor} \left[\binom{J+P-1}{k} \frac{(k+2)!}{(k+2)^\alpha} \frac{(J+P-k-2)!}{(J+P-k-2)^\alpha} \right] \right\}$$

and

$$\left\{ \dots \right\} \leq \frac{(J+P)^\alpha}{(J+P - \lfloor \frac{J+P}{2} \rfloor - 2)^\alpha} \frac{(J+P-1)!}{(J+P)!} \sum_{k=0}^{\lfloor \frac{J+P}{2} \rfloor} \frac{(k+2)(k+1)}{(J+P-k-1)(k+2)^\alpha} \leq C(\alpha) \quad , (16.24)$$

provided $\alpha \geq 1$. The estimate of the second term is slightly more delicate

$$\sum_{k=0}^{\lfloor \frac{J+P}{2} \rfloor} \sum_{h=0}^{p_J-1} \chi \left(k \leq (J+P+h-1-p_J) \right) \binom{J+P-1-(p_J-h)}{k} \left[\frac{(k+1)!}{(k+1)^\alpha} \frac{e^{(k-1)(\delta+\Gamma)} }{\rho_2^{k+1}} \frac{(J+P-k-1)!}{(J+P-k-1)^\alpha} \frac{e^{(J+P-k-3)(\delta+\Gamma(\lambda))}}{\rho_2^{J+P-k-1}} \right] \\ \leq c \left(\frac{(J+P)!}{(J+P)^\alpha} \frac{e^{(J+P-2)(\delta+\Gamma)}}{\rho_2^{J+P}} \right) \left(e^{-2(\delta+\Gamma(\lambda))} \right) \left\{ \sum_{k=0}^{\lfloor \frac{J+P}{2} \rfloor} \sum_{h=0}^{p_J-1} \chi \left(k \leq (J+P+h-1-p_J) \right) \cdot \right. \\ \left. \cdot \frac{(J+P)^\alpha}{(J+P)!} \binom{J+P-1-(p_J-h)}{k} \left[\frac{(k+1)!}{(k+1)^\alpha} \frac{(J+P-k-1)!}{(J+P-k-1)^\alpha} \right] \right\} \quad (16.25)$$

Observe that

$$\left\{ \dots \right\} \leq \left\{ \sum_{k=0}^{\lfloor \frac{J+P}{2} \rfloor} \frac{(J+P)^\alpha}{(J+P)!} \binom{J+P-1}{k+1} \left[\frac{(k+1)!}{(k+1)^\alpha} \frac{(J+P-k-1)!}{(J+P-k-1)^\alpha} \right] \right\} \\ \leq \frac{(J+P)^\alpha}{(J+P)!} \sum_{k=0}^{\lfloor \frac{J+P}{2} \rfloor} \frac{(J+P-1)!}{(J+P-k-2)!(k+1)!} \left[\frac{(k+1)!}{(k+1)^\alpha} \frac{(J+P-k-1)!}{(J+P-k-1)^\alpha} \right] \\ \leq \sum_{k=0}^{\lfloor \frac{J+P}{2} \rfloor} \frac{1}{(k+1)^\alpha} \left[\frac{(J+P)^\alpha (J+P-1)!(J+P-k-1)!}{(J+P-k-1)^\alpha (J+P)!(J+P-k-2)!} \right] \leq c(\alpha) \sum_{k=0}^{\lfloor \frac{J+P}{2} \rfloor} \frac{1}{(k+1)^\alpha} \leq C(\alpha) \quad . \quad (16.26)$$

provided $\alpha > 1$. The estimate of the sum with $k > \lfloor \frac{J+P}{2} \rfloor$ is done in the same way and has already been done in previous cases and we do not repeat here.

16.1 Proof of Lemma 3.6

We write, with $\sum_{s=0}^J p_s = P$,

$$\begin{aligned}
& \nabla^{p_0} \mathfrak{D}_\nu \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi \\
&= \mathfrak{D}_\nu \nabla^{p_0+p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi + [\nabla^{p_0}, \mathfrak{D}_\nu] \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi \\
&= \mathfrak{D}_\nu^2 \nabla^{p_0+p_1+p_2} \mathfrak{D}_\nu \nabla^{p_3} \mathfrak{D}_\nu \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi + \mathfrak{D}_\nu [\nabla^{p_0+p_1}, \mathfrak{D}_\nu] \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi \\
&\quad + [\nabla^{p_0}, \mathfrak{D}_\nu] \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi \\
&= \mathfrak{D}_\nu^3 \nabla^{p_0+p_1+p_2+p_3} \mathfrak{D}_\nu \nabla^{p_4} \mathfrak{D}_\nu \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi + \mathfrak{D}_\nu^2 [\nabla^{p_0+p_1+p_2}, \mathfrak{D}_\nu] \nabla^{p_3} \mathfrak{D}_\nu \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi \\
&\quad + \mathfrak{D}_\nu [\nabla^{p_0+p_1}, \mathfrak{D}_\nu] \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi + [\nabla^{p_0}, \mathfrak{D}_\nu] \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi \\
&= \mathfrak{D}_\nu^4 \nabla^{p_0+p_1+p_2+p_3+p_4} \mathfrak{D}_\nu \nabla^{p_5} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi + \mathfrak{D}_\nu^3 [\nabla^{p_0+p_1+p_2+p_3}, \mathfrak{D}_\nu] \nabla^{p_4} \mathfrak{D}_\nu \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi \\
&\quad + \mathfrak{D}_\nu^2 [\nabla^{p_0+p_1+p_2}, \mathfrak{D}_\nu] \nabla^{p_3} \mathfrak{D}_\nu \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi + \mathfrak{D}_\nu [\nabla^{p_0+p_1}, \mathfrak{D}_\nu] \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi \\
&\quad + [\nabla^{p_0}, \mathfrak{D}_\nu] \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi \\
&\quad \dots \dots \dots \\
&= \mathfrak{D}_\nu^J \nabla^{p_0+p_1+p_2+\dots+p_{J-1}+p_J} \Psi + \mathfrak{D}_\nu^{J-1} [\nabla^{p_0+p_1+p_2+\dots+p_{J-1}}, \mathfrak{D}_\nu] \nabla^{p_J} \Psi + \mathfrak{D}_\nu^{J-2} [\nabla^{p_0+p_1+p_2+\dots+p_{J-2}}, \mathfrak{D}_\nu] \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi \\
&\quad + \mathfrak{D}_\nu^{J-3} [\nabla^{p_0+p_1+p_2+\dots+p_{J-3}}, \mathfrak{D}_\nu] \nabla^{p_{J-2}} \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi + \mathfrak{D}_\nu^{J-4} [\nabla^{p_0+p_1+p_2+\dots+p_{J-4}}, \mathfrak{D}_\nu] \nabla^{p_{J-3}} \mathfrak{D}_\nu \nabla^{p_{J-2}} \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi \\
&\quad + \dots \dots \dots \\
&\quad + \mathfrak{D}_\nu^3 [\nabla^{p_0+p_1+p_2+p_3}, \mathfrak{D}_\nu] \nabla^{p_4} \mathfrak{D}_\nu \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi + \mathfrak{D}_\nu^2 [\nabla^{p_0+p_1+p_2}, \mathfrak{D}_\nu] \nabla^{p_3} \mathfrak{D}_\nu \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi \\
&\quad + \mathfrak{D}_\nu [\nabla^{p_0+p_1}, \mathfrak{D}_\nu] \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi + [\nabla^{p_0}, \mathfrak{D}_\nu] \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi \\
&\quad \dots \dots \dots \\
&= \mathfrak{D}_\nu^J \nabla^P \Psi + \sum_{t=1}^J \mathfrak{D}_\nu^{J-t} [\nabla^{\sum_{s=0}^{J-t} p_s}, \mathfrak{D}_\nu] \nabla^{p_{J-t+1}} \mathfrak{D}_\nu \nabla^{p_{J-t+2}} \mathfrak{D}_\nu \dots \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi
\end{aligned} \tag{16.27}$$

Therefore the final result is

$$\begin{aligned}
& \nabla^{p_0} \mathfrak{D}_\nu \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi \\
&= \mathfrak{D}_\nu^J \nabla^P \Psi + \sum_{t=1}^J \mathfrak{D}_\nu^{J-t} [\nabla^{\sum_{s=0}^{J-t} p_s}, \mathfrak{D}_\nu] \nabla^{p_{J-t+1}} \mathfrak{D}_\nu \nabla^{p_{J-t+2}} \mathfrak{D}_\nu \dots \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi.
\end{aligned} \tag{16.28}$$

and we have the following estimate,

$$\begin{aligned}
& \left| |\lambda|_-^{\phi(\Psi)} r^{1+J+P+\phi(\Psi)-\frac{2}{p}} \nabla^{p_0} \mathfrak{D}_\nu \nabla^{p_1} \mathfrak{D}_\nu \nabla^{p_2} \dots \mathfrak{D}_\nu \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi \right|_{p,S} \leq \left| |\lambda|_-^{\phi(\Psi)} r^{1+J+P+\phi(\Psi)-\frac{2}{p}} \mathfrak{D}_\nu^J \nabla^P \Psi \right|_{p,S} \\
& + \sum_{t=1}^J \left| |\lambda|_-^{\phi(\Psi)} r^{1+J+P+\phi(\Psi)-\frac{2}{p}} \mathfrak{D}_\nu^{J-t} [\nabla^{\sum_{s=0}^{J-t} p_s}, \mathfrak{D}_\nu] \nabla^{p_{J-t+1}} \mathfrak{D}_\nu \nabla^{p_{J-t+2}} \mathfrak{D}_\nu \dots \nabla^{p_{J-1}} \mathfrak{D}_\nu \nabla^{p_J} \Psi \right|_{p,S}.
\end{aligned}$$

We make the following inductive assumptions, for $J + P < N - 1$,

$$||\lambda|_{\underline{\omega}}^{\phi(\Psi)} r^{1+J+P+\phi(\Psi)-\frac{2}{p}} \nabla^{p_0} \mathbf{p}_\nu \nabla^{p_1} \mathbf{p}_\nu \nabla^{p_2} \dots \mathbf{p}_\nu \nabla^{p_{J-1}} \mathbf{p}_\nu \nabla^{p_J} \Psi|_{p,S} \leq F \left(\frac{(J+P+1)!}{(J+P+1)^\alpha} \frac{e^{(J+P-1)(\delta+\Gamma(\lambda))}}{\rho_2^{J+P+1}} \right) . \quad (16.29)$$

and use the following expression for the commutator

$$\begin{aligned} [\nabla^{p_J}, \mathbf{p}_\nu] \Psi &= - \sum_{q=0}^{p_J-1} (p_J - q - 1) \sum_{h=0}^q \binom{q}{h} (\nabla^h C) \nabla^{p_J-1-h} U + \sum_{q=0}^{p_J-1} \sum_{h=0}^q \binom{q}{h} (\nabla^h \chi) \nabla^{p_J-h} \Psi \\ &= - \sum_{h=0}^{p_J-2} \binom{p_J}{p_J-2-h} (\nabla^h C) \nabla^{p_J-1-h} U + \sum_{q=0}^{p_J-1} \sum_{h=0}^q \binom{q}{h} (\nabla^h \chi) \nabla^{p_J-h} \Psi \\ &= - \sum_{h=0}^{p_J-2} \binom{p_J}{p_J-2-h} (\nabla^h C) \nabla^{p_J-1-h} \Psi + \sum_{h=0}^{p_J-1} \binom{p_J}{p_J-1-h} (\nabla^h \chi) \nabla^{p_J-h} \Psi \\ &= - \sum_{q=0}^{p_J-2} \binom{p_J}{q} (\nabla^{p_J-2-q} C) \nabla^{q+1} \Psi + \sum_{q=0}^{p_J-1} \binom{p_J}{q} (\nabla^{p_J-1-q} \chi) \nabla^{q+1} \Psi \end{aligned} \quad (16.30)$$

where

$$C \equiv C_{\mu\nu}^\sigma = (\chi_{\mu\nu} \underline{\eta}^\sigma - \chi_\mu^\sigma \underline{\eta}_\nu) + \theta_\mu^C \theta_\nu^D R^\sigma(\cdot, e_C, e_4, e_D) . \quad (16.31)$$

We start the estimate,

$$\begin{aligned} &\sum_{t=1}^J ||\lambda|_{\underline{\omega}}^{\phi(\Psi)} r^{1+J+P+\phi(\Psi)-\frac{2}{p}} \mathbf{p}_\nu^{J-t} [\nabla(\sum_{s=0}^{J-t} p_s), \mathbf{p}_\nu] \nabla^{p_{J-t+1}} \mathbf{p}_\nu \nabla^{p_{J-t+2}} \mathbf{p}_\nu \dots \nabla^{p_{J-1}} \mathbf{p}_\nu \nabla^{p_J} \Psi|_{p,S} \\ &\leq \sum_{t=1}^J \sum_{h=0}^{(\sum_{s=0}^{J-t} p_s)-2} \binom{(\sum_{s=0}^{J-t} p_s)}{(\sum_{s=0}^{J-t} p_s) - 2 - h} \cdot ||\lambda|_{\underline{\omega}}^{\phi(\Psi)} r^{1+J+P+\phi(\Psi)-\frac{2}{p}} \mathbf{p}_\nu^{J-t} (\nabla^h C) (\nabla(\sum_{s=0}^{J-t} p_s - 1 - h) \nabla^{p_{J-t+1}} \mathbf{p}_\nu \nabla^{p_{J-t+2}} \mathbf{p}_\nu \dots \nabla^{p_{J-1}} \mathbf{p}_\nu \nabla^{p_J} \Psi)|_{p,S} \\ &+ \sum_{t=1}^J \sum_{h=0}^{(\sum_{s=0}^{J-t} p_s)-1} \binom{(\sum_{s=0}^{J-t} p_s)}{(\sum_{s=0}^{J-t} p_s) - 1 - h} \cdot ||\lambda|_{\underline{\omega}}^{\phi(\Psi)} r^{1+J+P+\phi(\Psi)-\frac{2}{p}} \mathbf{p}_\nu^{J-t} (\nabla^h \chi) (\nabla(\sum_{s=0}^{J-t} p_s - h) \nabla^{p_{J-t+1}} \mathbf{p}_\nu \nabla^{p_{J-t+2}} \mathbf{p}_\nu \dots \nabla^{p_{J-1}} \mathbf{p}_\nu \nabla^{p_J} \Psi)|_{p,S} . \end{aligned}$$

Recalling previous definitions we define

$$\sum_{s=0}^J p_s = P \quad , \quad \sum_{s=0}^{J-t} p_s = P_{(J-t)}, \quad J + P = N ,$$

and we estimate the second sum as

$$|(II)|_{p,S} \leq \quad (16.32)$$

$$\begin{aligned}
&\leq \sum_{t=1}^J \sum_{h=0}^{P_{(J-t)}-1} \sum_{l=0}^{J-t} \binom{P_{(J-t)}}{P_{(J-t)}-1-h} \binom{J-t}{l} \cdot \\
&\quad \cdot \left| |\lambda|_{-}^{\phi(\Psi)} r^{1+J+P+\phi(\Psi)-\frac{2}{p}} (\mathfrak{D}_{\nu}^l \nabla^h \chi) (\mathfrak{D}_{\nu}^{J-t-l} \nabla^{(P_{(J-t+1)}-h)} \mathfrak{D}_{\nu} \nabla^{p_{J-t+2}} \mathfrak{D}_{\nu} \cdot \cdot \nabla^{p_{J-1}} \mathfrak{D}_{\nu} \nabla^{p_J} \Psi) \right|_{p,S} \\
&\leq \sum_{t=1}^J \sum_{h=0}^{P_{(J-t)}-1} \sum_{q=0}^{J-t} \binom{P_{(J-t)}}{h+1} \binom{J-t}{q} \chi \left((q+h) \leq \left\lfloor \frac{J+P}{2} \right\rfloor \right) \cdot \\
&\quad \left| |\lambda|_{-}^{\phi(\Psi)} r^{1+J+P+\phi(\Psi)-\frac{2}{p}} (\mathfrak{D}_{\nu}^q \nabla^h \chi) (\mathfrak{D}_{\nu}^{J-t-q} \nabla^{(P_{(J-t+1)}-h)} \mathfrak{D}_{\nu} \nabla^{p_{J-t+2}} \mathfrak{D}_{\nu} \cdot \cdot \nabla^{p_{J-1}} \mathfrak{D}_{\nu} \nabla^{p_J} \Psi) \right|_{p,S} \\
&+ \sum_{t=1}^J \sum_{h=0}^{P_{(J-t)}-1} \sum_{q=0}^{J-t} \binom{P_{(J-t)}}{h+1} \binom{J-t}{q} \chi \left((q+h) > \left\lfloor \frac{J+P}{2} \right\rfloor \right) \cdot \\
&\quad \left| |\lambda|_{-}^{\phi(\Psi)} r^{1+J+P+\phi(\Psi)-\frac{2}{p}} (\mathfrak{D}_{\nu}^q \nabla^h \chi) (\mathfrak{D}_{\nu}^{J-t-q} \nabla^{(P_{(J-t+1)}-h)} \mathfrak{D}_{\nu} \nabla^{p_{J-t+2}} \mathfrak{D}_{\nu} \cdot \cdot \nabla^{p_{J-1}} \mathfrak{D}_{\nu} \nabla^{p_J} \Psi) \right|_{p,S} .
\end{aligned}$$

Let us consider the first sum

$$\begin{aligned}
|(II)_1|_{p,S} &\leq \sum_{t=1}^J \sum_{h=0}^{P_{(J-t)}-1} \sum_{q=0}^{J-t} \binom{P_{(J-t)}}{h+1} \binom{J-t}{q} \chi \left((q+h) \leq \left\lfloor \frac{J+P}{2} \right\rfloor \right) \cdot \\
&\quad \left| r^{1+q+h} (\mathfrak{D}_{\nu}^q \nabla^h \chi) \right|_{\infty,S} \left| |\lambda|_{-}^{\phi(\Psi)} r^{1+J+P-(q+h)-t+\phi(\Psi)-\frac{2}{p}} (\mathfrak{D}_{\nu}^{J-t-q} \nabla^{(P_{(J-t+1)}-h)} \mathfrak{D}_{\nu} \nabla^{p_{J-t+2}} \mathfrak{D}_{\nu} \cdot \cdot \nabla^{p_{J-1}} \mathfrak{D}_{\nu} \nabla^{p_J} \Psi) \right|_{p,S} \\
&\leq \sum_{t=1}^J \sum_{h=0}^{P_{(J-t)}-1} \sum_{q=0}^{J-t} \binom{P_{(J-t)}}{h+1} \binom{J-t}{q} \chi \left((q+h) \leq \left\lfloor \frac{J+P}{2} \right\rfloor \right) \\
&\quad \left(\frac{(q+h+1)!}{(q+h+1)^\alpha} \frac{e^{(q+h-1)(\delta+\Gamma(\lambda))}}{\rho_2^{q+h+1}} \right) \cdot F \left(\frac{(J+P+1-(q+h)-t)!}{(J+P+1-(q+h)-t)^\alpha} \frac{e^{(J+P-(q+h)-t-1)(\delta+\Gamma(\lambda))}}{\rho_2^{J+P-(q+h)-t+1}} \right) \\
&\leq f \left(\frac{(J+P+1)!}{(J+P+1)^\alpha} \frac{e^{(J+P-1)(\delta+\Gamma(\lambda))}}{\rho_2^{J+P+1}} \right) \sum_{t=1}^J \left(\frac{e^{-(t+1)(\delta+\Gamma(\lambda))}}{\rho_2^{-t+1}} \right) \left\{ \frac{(J+P+1)^\alpha}{(J+P+1)!} \right. \\
&\quad \left. \left[\sum_{h=0}^{P_{(J-t)}-1} \sum_{q=0}^{J-t} \binom{P_{(J-t)}}{h+1} \binom{J-t}{q} \chi \left((q+h) \leq \left\lfloor \frac{J+P}{2} \right\rfloor \right) \left(\frac{(q+h+1)!}{(q+h+1)^\alpha} \right) \cdot \left(\frac{(J+P+1-(q+h)-t)!}{(J+P+1-(q+h)-t)^\alpha} \right) \right] \right\} .
\end{aligned}$$

Let us denote $(q+h) = k$ then, observing that, denoting $\hat{h} = h+1$, $\hat{k} = k+1$

$$\begin{aligned}
&\sum_{h=0}^{P_{(J-t)}-1} \sum_{q=0}^{J-t} \binom{P_{(J-t)}}{h+1} = \sum_{\hat{h}=1}^{P_{(J-t)}} \sum_{q=0}^{J-t} \binom{P_{(J-t)}}{\hat{h}} \binom{J-t}{q} \text{ " } \leq \text{ " } \sum_{\hat{h}=0}^{P_{(J-t)}} \sum_{q=0}^{J-t} \binom{P_{(J-t)}}{\hat{h}} \binom{J-t}{q} \\
&\leq \sum_{\hat{k}=0}^{P_{(J-t)}+J-t} \binom{P_{(J-t)}+J-t}{\hat{k}} \tag{16.33}
\end{aligned}$$

$$\left[\cdot \right] \leq \left[\sum_{\hat{k}=0}^{\left\lfloor \frac{J+P}{2} \right\rfloor} \binom{P_{(J-t)}+J-t}{\hat{k}} \frac{\hat{k}!}{\hat{k}^\alpha} \frac{(J+P+2-\hat{k}-t)!}{(J+P+2-\hat{k}-t)^\alpha} \right]$$

and

$$\begin{aligned}
& \left\{ \frac{(J+P+1)^\alpha}{(J+P+1)!} \cdot \left[\sum_{h=0}^{P(J-t)-1} \sum_{q=0}^{J-t} \binom{P(J-t)}{h+1} \binom{J-t}{q} \chi\left((q+h) \leq \left\lfloor \frac{J+P}{2} \right\rfloor\right) \left(\frac{(q+h+1)!}{(q+h+1)^\alpha} \right) \cdot \left(\frac{(J+P+1-(q+h)-t)!}{(J+P+1-(q+h)-t)^\alpha} \right) \right] \right\} \\
& \leq \frac{(J+P+1)^\alpha}{(J+P+1)!} \left[\sum_{\hat{k}=0}^{\left\lfloor \frac{J+P}{2} \right\rfloor \wedge (J+P(J-t))} \binom{P(J-t)+J-t}{\hat{k}} \frac{\hat{k}!}{\hat{k}^\alpha} \frac{(J+P+2-\hat{k}-t)!}{(J+P+2-\hat{k}-t)^\alpha} \right] \\
& \leq (J+P+1)^\alpha \left[\sum_{\hat{k}=0}^{\left\lfloor \frac{J+P}{2} \right\rfloor \wedge (J+P(J-t))} \frac{1}{\hat{k}^\alpha} \frac{(J+P(J-t)-t)!(J+P+2-\hat{k}-t)!}{(J+P(J-t)-\hat{k}-t)!(J+P+1)!} \frac{1}{(J+P+2-\hat{k}-t)^\alpha} \right] \\
& \leq C(\alpha) \frac{(J+P+1)^\alpha}{\left(\frac{J+P}{2}-t\right)^\alpha}, \tag{16.34}
\end{aligned}$$

provided, $\alpha > 1$. Therefore

$$\begin{aligned}
& \sum_{t=1}^J \left(\frac{e^{-(t+1)(\delta+\Gamma)}}{\rho^{-t}} \right) \left\{ \frac{(J+P+2)^\alpha}{(J+P+2)!} \right. \\
& \left[\sum_{h=0}^{P(J-t)-1} \sum_{q=0}^{J-t} \binom{P(J-t)}{h+1} \binom{J-t}{q} \chi\left((q+h) \leq \left\lfloor \frac{J+P}{2} \right\rfloor\right) \left(\frac{(q+h+2)!}{(q+h+2)^\alpha} \right) \cdot \left(\frac{(J+P+2-(q+h)-t)!}{(J+P+2-(q+h)-t)^\alpha} \right) \right] \right\} \\
& \leq e^{-\delta} C(\alpha) \sum_{t=1}^J \left(\rho^t e^{-t(\delta+\Gamma)} \right) \frac{(J+P+1)^\alpha}{\left(\frac{J+P}{2}-t\right)^\alpha} \leq e^{-\delta} C_1(\alpha), \tag{16.35}
\end{aligned}$$

completing the proof of the estimate.¹³⁴ Therefore we have proved that the following estimate holds with $J+P \leq N-1$,

$$\| |\lambda|^{\phi(\Psi)} r^{1+J+P+\phi(\Psi)-\frac{2}{p}} \nabla^{p_0} \mathbf{D}_\nu \nabla^{p_1} \mathbf{D}_\nu \nabla^{p_2} \dots \mathbf{D}_\nu \nabla^{p_{J-1}} \mathbf{D}_\nu \nabla^{p_J} \Psi \|_{p,S} \leq F_1 \left(\frac{(J+P+1)!}{(J+P+1)^\alpha} \frac{e^{(J+P-1)(\delta+\Gamma(\lambda))}}{\rho_2^{J+P+1}} \right), \tag{16.36}$$

provided that the following estimate holds with $F_2 < F_1$,

$$\| |\lambda|^{\phi(\Psi)} r^{1+J+P+\phi(\Psi)-\frac{2}{p}} \mathbf{D}_\nu^J \nabla^P \Psi \|_{p,S} \leq F_2 \left(\frac{(J+P+1)!}{(J+P+1)^\alpha} \frac{e^{(J+P-1)(\delta+\Gamma(\lambda))}}{\rho_2^{J+P+1}} \right). \tag{16.37}$$

¹³⁴There are, in fact, other sums to estimate, but it is easy to realize that all the estimates go in the same way.

17 Appendix to Section 11

17.1 Proof of the estimate 11.3

We have to control the right hand side of the following expression,

$$|\nabla^N \Omega|_{p,S} \leq N! \sum_{k=1}^{\infty} \frac{1}{k!} \left[\sum_{\gamma_1=0}^{\frac{N}{2}} \sum_{\gamma_2, \dots, \gamma_k}^{\sum_{s=2}^k \gamma_s = N - \gamma_1; \gamma_s \in [0, N]} \frac{1}{\gamma_1! \gamma_2! \dots \gamma_k!} |(\nabla^{\gamma_1} \log \Omega)(\nabla^{\gamma_2} \log \Omega) \dots (\nabla^{\gamma_k} \log \Omega)|_{p,S} \right. \\ \left. \sum_{\gamma_1=\frac{N}{2}+1}^N \sum_{\gamma_2, \dots, \gamma_k}^{\sum_{s=2}^k \gamma_s = N - \gamma_1; \gamma_s \in [0, N]} \frac{1}{\gamma_1! \gamma_2! \dots \gamma_k!} |(\nabla^{\gamma_1} \log \Omega)(\nabla^{\gamma_2} \log \Omega) \dots (\nabla^{\gamma_k} \log \Omega)|_{p,S} \right].$$

Let us consider the second sum

$$\begin{aligned} & N! \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\gamma_1=\frac{N}{2}+1}^N \sum_{\gamma_2, \dots, \gamma_k}^{\sum_{s=2}^k \gamma_s = N - \gamma_1; \gamma_s \in [0, N]} \frac{1}{\gamma_1! \gamma_2! \dots \gamma_k!} |(\nabla^{\gamma_1} \log \Omega)(\nabla^{\gamma_2} \log \Omega) \dots (\nabla^{\gamma_k} \log \Omega)|_{p,S} \\ & \leq N! \sum_{\gamma_1=\frac{N}{2}+1}^N \frac{|(\nabla^{\gamma_1} \log \Omega)|_{p,S}}{\gamma_1!} \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\gamma_2, \dots, \gamma_k}^{\sum_{s=2}^k \gamma_s = N - \gamma_1; \gamma_s \in [0, N]} \frac{1}{\gamma_2! \dots \gamma_k!} |(\nabla^{\gamma_2} \log \Omega)|_{\infty, S} \dots |(\nabla^{\gamma_k} \log \Omega)|_{\infty, S} \\ & \leq N! \sum_{\gamma_1=\frac{N}{2}+1}^N \tilde{C}_3 \frac{(\gamma_1 - 1)!}{(\gamma_1 - 1)^\alpha \gamma_1!} \frac{e^{(\gamma_1 - 3)(\delta + \Gamma(\lambda))}}{\rho^{\gamma_1 - 1}} \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\gamma_2, \dots, \gamma_k}^{\sum_{s=2}^k \gamma_s = N - \gamma_1; \gamma_s \in [0, N]} \frac{\tilde{C}_3^{\sigma(\gamma_2) + \dots + \sigma(\gamma_k)}}{\gamma_2^\alpha \dots \gamma_k^\alpha} \frac{e^{\sigma(\gamma_2 - 2)(\gamma_2 - 2)(\delta + \Gamma(\lambda))}}{\rho^{\gamma_2}} \dots \frac{e^{\sigma(\gamma_k - 2)(\gamma_k - 2)(\delta + \Gamma(\lambda))}}{\rho^{\gamma_k}} \\ & \leq \left(\tilde{C}_3 \frac{(N - 1)!}{(N - 1)^\alpha} \frac{e^{(N - 3)(\delta + \Gamma(\lambda))}}{\rho^{N - 1}} \right) N \sum_{\gamma_1=\frac{N}{2}+1}^N \frac{(N - 1)^\alpha}{(\gamma_1 - 1)^\alpha \gamma_1} \frac{e^{-(N - \gamma_1)(\delta + \Gamma(\lambda))}}{\rho^{-(N - \gamma_1)}} \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\gamma_2, \dots, \gamma_k}^{\sum_{s=2}^k \gamma_s = N - \gamma_1; \gamma_s \in [0, N]} \frac{1}{\gamma_2^\alpha \dots \gamma_k^\alpha} \frac{1}{\rho^{(N - \gamma_1)}} \\ & \leq \left(\tilde{C}_3 \frac{(N - 1)!}{(N - 1)^\alpha} \frac{e^{(N - 3)(\delta + \Gamma(\lambda))}}{\rho^{N - 1}} \right) \sum_{\gamma_1=\frac{N}{2}+1}^N \frac{N^{\alpha+1}}{\gamma_1^{\alpha+1}} e^{-(N - \gamma_1)(\delta + \Gamma(\lambda))} \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\gamma_2, \dots, \gamma_k}^{\sum_{s=2}^k \gamma_s = N - \gamma_1; \gamma_s \in [0, N]} \frac{1}{\gamma_2^\alpha \dots \gamma_k^\alpha} \\ & \leq \left(\tilde{C}_3 \frac{(N - 1)!}{(N - 1)^\alpha} \frac{e^{(N - 3)(\delta + \Gamma(\lambda))}}{\rho^{N - 1}} \right) \left[\sum_{\gamma_1=\frac{N}{2}+1}^N \frac{N^{\alpha+1}}{\gamma_1^{\alpha+1}} e^{-(N - \gamma_1)(\delta + \Gamma(\lambda))} \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\gamma_2, \dots, \gamma_k}^{\sum_{s=2}^k \gamma_s = N - \gamma_1; \gamma_s \in [0, N]} \frac{1}{\gamma_2^\alpha \dots \gamma_k^\alpha} \right]. \end{aligned}$$

Let us consider the first sum which we rewrite as

$$\begin{aligned} & N! \sum_{k=1}^{\infty} \frac{1}{k!} \left[\sum_{\gamma_2=0}^{\frac{N}{2}} \sum_{\gamma_1=0}^{\frac{N}{2}} \sum_{\gamma_3, \dots, \gamma_k}^{\sum_{s=2}^k \gamma_s = N - \gamma_1; \gamma_s \in [0, N]} \frac{1}{\gamma_1! \gamma_2! \dots \gamma_k!} |(\nabla^{\gamma_1} \log \Omega)(\nabla^{\gamma_2} \log \Omega) \dots (\nabla^{\gamma_k} \log \Omega)|_{p,S} \right. \\ & \quad \left. \sum_{\gamma_1=\frac{N}{2}+1}^N \sum_{\gamma_1=0}^{\frac{N}{2}} \sum_{\gamma_3, \dots, \gamma_k}^{\sum_{s=2}^k \gamma_s = N - \gamma_1; \gamma_s \in [0, N]} \frac{1}{\gamma_1! \gamma_2! \dots \gamma_k!} |(\nabla^{\gamma_1} \log \Omega)(\nabla^{\gamma_2} \log \Omega) \dots (\nabla^{\gamma_k} \log \Omega)|_{p,S} \right] \quad (17.2) \end{aligned}$$

Let us consider the second sum of the first sum which we estimate as

$$\begin{aligned}
&\leq N! \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\gamma_2=\frac{N}{2}+1}^N \frac{|(\nabla^{\gamma_2} \log \Omega)|_{p,S}}{\gamma_2!} \sum_{\gamma_1=0}^{\frac{N}{2}} \sum_{\gamma_3, \dots, \gamma_k}^{\gamma_s=N-\gamma_1-\gamma_2; \gamma_s \in [0, N]} \frac{1}{\gamma_1! \gamma_3! \dots \gamma_k!} |(\nabla^{\gamma_1} \log \Omega)|_{\infty, S} |(\nabla^{\gamma_3} \log \Omega)|_{\infty, S} \dots |(\nabla^{\gamma_k} \log \Omega)|_{\infty, S} \\
&\leq N! \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\gamma_2=\frac{N}{2}+1}^N \frac{|(\nabla^{\gamma_2} \log \Omega)|_{p,S}}{\gamma_2!} \sum_{s=1}^{\sum_{s=1}^{k; s \neq 2} \gamma_s=N-\gamma_2; \gamma_s \in [0, N]} \frac{1}{\gamma_1! \gamma_3! \dots \gamma_k!} |(\nabla^{\gamma_1} \log \Omega)|_{\infty, S} |(\nabla^{\gamma_3} \log \Omega)|_{\infty, S} \dots |(\nabla^{\gamma_k} \log \Omega)|_{\infty, S} \quad (17.3)
\end{aligned}$$

The last sum is identically to the previous “second sum” interchanging γ_1 and γ_2 and therefore can be bounded by

$$\leq \left(\tilde{C}_3 \frac{(N-1)!}{(N-1)^\alpha} \frac{e^{(N-3)(\delta+\Gamma(\lambda))}}{\rho^{N-1}} \right) \left[\sum_{\gamma_1=\frac{N}{2}+1}^N \frac{N^{\alpha+1}}{\gamma_1^{\alpha+1}} e^{-(N-\gamma_1)(\delta+\Gamma(\lambda))} \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\gamma_2, \dots, \gamma_k}^{\gamma_s=N-\gamma_1; \gamma_s \in [0, N]} \frac{1}{\gamma_2^\alpha \dots \gamma_k^\alpha} \right] \quad (17.4)$$

as this has to be repeated k times obtaining

$$\begin{aligned}
&\leq \left(\tilde{C}_3 \frac{(N-1)!}{(N-1)^\alpha} \frac{e^{(N-3)(\delta+\Gamma(\lambda))}}{\rho^{N-1}} \right) \left[\sum_{\gamma_1=\frac{N}{2}+1}^N \frac{N^{\alpha+1}}{\gamma_1^{\alpha+1}} e^{-(N-\gamma_1)(\delta+\Gamma(\lambda))} \sum_{k=1}^{\infty} \frac{k}{k!} \sum_{\gamma_2, \dots, \gamma_k}^{\gamma_s=N-\gamma_1; \gamma_s \in [0, N]} \frac{1}{\gamma_2^\alpha \dots \gamma_k^\alpha} \right] \\
&\leq \left(\tilde{C}_3 \frac{(N-1)!}{(N-1)^\alpha} \frac{e^{(N-3)(\delta+\Gamma(\lambda))}}{\rho^{N-1}} \right) \left[\sum_{\gamma_1=\frac{N}{2}+1}^N \frac{N^{\alpha+1}}{\gamma_1^{\alpha+1}} e^{-(N-\gamma_1)(\delta+\Gamma(\lambda))} \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \sum_{\gamma_2, \dots, \gamma_k}^{\gamma_s=N-\gamma_1; \gamma_s \in [0, N]} \frac{1}{\gamma_2^\alpha \dots \gamma_k^\alpha} \right] \\
&\leq \left(\tilde{C}_3 \frac{(N-1)!}{(N-1)^\alpha} \frac{e^{(N-3)(\delta+\Gamma(\lambda))}}{\rho^{N-1}} \right) (1 + ce^{-\delta}) . \quad (17.5)
\end{aligned}$$

We are left with a term where all the γ_i are $\leq \left[\frac{N}{2}\right]$ which is easier to estimate so that the final result is

$$|\nabla^N \Omega|_{p,S} \leq \left(\tilde{C}_3 \frac{(N-1)!}{(N-1)^\alpha} \frac{e^{(N-3)(\delta+\Gamma(\lambda))}}{\rho^{N-1}} \right) (1 + ce^{-\delta}) . \quad (17.6)$$

As $\partial_\nu \Omega = \Omega \mathfrak{P}_4 \Omega = -2\Omega \omega$, to control $\partial_\nu^N \Omega$ we write

$$\begin{aligned}
\mathfrak{P}_\nu^N \Omega &= \mathfrak{P}_\nu^N e^{\log \Omega} = \sum_{k=0}^{\infty} \frac{1}{k!} \mathfrak{P}_\nu^N (\log \Omega)^k \\
&= N! \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\gamma_1, \dots, \gamma_k}^{\gamma_s=N; \gamma_s \in [0, N]} \frac{1}{\gamma_1! \gamma_2! \dots \gamma_k!} (\mathfrak{P}_\nu^{\gamma_1} \log \Omega) (\mathfrak{P}_\nu^{\gamma_2} \log \Omega) \dots (\mathfrak{P}_\nu^{\gamma_k} \log \Omega) \\
&= N! \sum_{k=1}^{\infty} \frac{(-2)^k}{k!} \sum_{\gamma_1, \dots, \gamma_k}^{\gamma_s=N; \gamma_s \in [0, N]} \frac{1}{\gamma_1! \gamma_2! \dots \gamma_k!} (\mathfrak{P}_\nu^{\gamma_1-1} \omega) (\mathfrak{P}_\nu^{\gamma_2-1} \omega) \dots (\mathfrak{P}_\nu^{\gamma_k-1} \omega) .
\end{aligned}$$

As the following estimates hold, see Theorem 3.4,

$$|r^{2+J-\frac{2}{p}} \mathbf{D}_4^J \omega|_{p,S}(\lambda, \nu) \leq E_5 \left(\frac{(J+1)!}{(J+1)^\alpha} \frac{e^{((J+1)-2)(\delta+\Gamma(\lambda))}}{\rho^{J+1}} \right), \quad (17.7)$$

proceeding as before it is easy to prove the following estimate

$$|\partial_\nu^N \Omega|_{p,S} \leq c \left(\frac{N!}{N^\alpha} \frac{e^{(N-2)(\delta+\Gamma(\lambda))}}{\rho^N} \right). \quad (17.8)$$

17.2 Proof of the estimate 11.15

From the transport equation

$$\frac{p}{\partial \nu} (\nabla^J |X|^2) + (J+2) \frac{\Omega \text{tr} \chi}{2} \cdot \nabla^J |X|^2 + J(\Omega \hat{\chi}) \cdot \nabla^J |X|^2 - 2(\nabla^J X) \cdot \Omega \hat{\chi} \cdot X + 2(\nabla^J X) \cdot \zeta = \left\{ good \right\}_X \quad (17.9)$$

where

$$\begin{aligned} \left\{ good \right\}_X &= - \left\{ \left[\sum_{k=0}^{J-2} \binom{J}{k} (\nabla^{J-1-k} \Omega \chi) \cdot \nabla^{k+1} |X|^2 - \sum_{k=0}^{J-1} \binom{J+1}{k} (\nabla^{J-1-k} C) \nabla^k |X|^2 \right] \right. \\ &\quad - \left[\sum_{k=1}^J \binom{J}{k} (\nabla^k \Omega \text{tr} \chi) \nabla^{J-k} |X|^2 + \sum_{\substack{s=1 \\ \gamma_1 \gamma_2 \gamma_3}}^4 \gamma_s = J; \gamma_1, \gamma_3 \neq J \frac{J!}{\gamma_1! \gamma_2! \gamma_3!} (\nabla^{\gamma_1} X) \cdot (\nabla^{\gamma_2} \Omega \hat{\chi}) \cdot (\nabla^{\gamma_3} X) \right. \\ &\quad \left. \left. + 2 \sum_{k=0}^{J-1} \sum_{l=0}^{J-k} \frac{J!}{k!(J-k)!} (\nabla^k X) \nabla^{J-k} \zeta \right] \right\}. \end{aligned}$$

it follows, as done many times before,

$$\begin{aligned} &\frac{\partial |\nabla^J |X|^2|^p}{\partial \nu} + p(J+2) \frac{\Omega \text{tr} \chi}{2} |\nabla^J |X|^2|^p \\ &= -p |\nabla^J |X|^2|^{p-2} \left[J(\nabla^J |X|^2) \cdot \Omega \hat{\chi} \cdot (\nabla^J |X|^2) + (\nabla^J |X|^2) (2(\nabla^J X) \cdot \Omega \hat{\chi} \cdot X - 2(\nabla^J X) \cdot \zeta) \right. \\ &\quad \left. + (\nabla^J |X|^2) \cdot \left\{ good \right\}_X \right] \\ &\leq p |\nabla^J |X|^2|^{p-1} J |\Omega \hat{\chi} \cdot (\nabla^J |X|^2)| + 2p |\nabla^J |X|^2|^{p-1} |\nabla^J X \cdot \Omega \hat{\chi} \cdot X| + 2p |\nabla^J |X|^2|^{p-1} |\nabla^J X \cdot \zeta| \\ &\quad + p |\nabla^J |X|^2|^{p-1} |\left\{ good \right\}_X| \end{aligned} \quad (17.10)$$

which we rewrite as

$$\begin{aligned} &\frac{\partial |\nabla^J |X|^2|^p}{\partial \nu} + p(J+2) \frac{\Omega \text{tr} \chi}{2} |\nabla^J |X|^2|^p \\ &\leq p |\nabla^J |X|^2|^{p-1} \left\{ \left[J |\Omega \hat{\chi} \cdot (\nabla^J |X|^2)| \right] + \left[2 \left(|\nabla^J X \cdot \Omega \hat{\chi} \cdot X| + |\nabla^J X \cdot \zeta| \right) + |\left\{ good \right\}_X| \right] \right\}. \end{aligned} \quad (17.11)$$

and in a more compact way as

$$\frac{\partial |\nabla^J |X|^2|^p}{\partial \nu} + p(J+2) \frac{\Omega \text{tr} \chi}{2} |\nabla^J |X|^2|^p \leq p |\nabla^J |X|^2|^{p-1} |L_X| . \quad (17.12)$$

where

$$|L_X| = J |\Omega \hat{\chi} \cdot (\nabla^J |X|^2)| + \left[2 \left(|\nabla^J X \cdot \Omega \hat{\chi} \cdot X| + |\nabla^J X \cdot \zeta| \right) + p |\nabla^J |X|^2|^{p-1} |\{good\}_X| \right] \quad (17.13)$$

From it,

$$\begin{aligned} \frac{\partial |r^{(J+2-\frac{2}{p})} \nabla^J |X|^2|_{p,S}^p}{\partial \nu} &= \int_S \left(\frac{\partial |r^{(J+2-\frac{2}{p})} \nabla^J |X|^2|^p}{\partial \nu} + \Omega \text{tr} \chi |r^{(J+2-\frac{2}{p})} \nabla^J |X|^2|^p \right) d\mu_\gamma \\ &= \int_S \left(r^{(J+2-\frac{2}{p})p} \left[\frac{\partial |\nabla^J |X|^2|^p}{\partial \nu} + p(J+2) \frac{\overline{\Omega \text{tr} \chi}}{2} |\nabla^J |X|^2|^p \right] + (\Omega \text{tr} \chi - \overline{\Omega \text{tr} \chi}) |r^{(J+2-\frac{2}{p})} \nabla^J |X|^2|^p \right) d\mu_\gamma \\ &\leq p \int_S r^{(J+2-\frac{2}{p})p} |\nabla^J |X|^2|^{p-1} \left[|L_X| + \left(\frac{J+2-\frac{2}{p}}{2} |\Omega \text{tr} \chi - \overline{\Omega \text{tr} \chi}| \right) |\nabla^J |X|^2| \right] , \end{aligned}$$

which we rewrite,

$$\frac{\partial |r^{(J+2-\frac{2}{p})} \nabla^J |X|^2|_{p,S}^p}{\partial \nu} \leq p \int_S r^{(J+2-\frac{2}{p})p} |\nabla^J |X|^2|^{p-1} |\mathcal{L}_X| d\mu_\gamma , \quad (17.14)$$

where

$$|\mathcal{L}_X| = \left[J (|\Omega \text{tr} \chi - \overline{\Omega \text{tr} \chi}| + |\Omega \hat{\chi}|) |\nabla^J |X|^2| + |\tilde{L}_X| \right] \quad (17.15)$$

and

$$\tilde{L}_X = \left[2 \left(|\nabla^J X \cdot \Omega \hat{\chi} \cdot X| + |\nabla^J X \cdot \zeta| \right) + |\{good\}_X| \right] . \quad (17.16)$$

Proceeding as before we have, denoting $\mathcal{H} = |\Omega \hat{\chi}| + |\overline{\Omega \text{tr} \chi}| + |\Omega \hat{\chi}|$,

$$\frac{\partial |r^{(J+2-\frac{2}{p})} \nabla^J |X|^2|_{p,S}}{\partial \nu} \leq |r^{(J+2-\frac{2}{p})} \nabla^J \mathcal{L}_X|_{p,S} \leq J |\mathcal{H}|_{\infty,S} |r^{(J+2-\frac{2}{p})} \nabla^J |X|^2|_{p,S} + |r^{(J+2-\frac{2}{p})} \tilde{L}_X|_{p,S}$$

and integrating

$$|r^{(J+2-\frac{2}{p})} \nabla^J |X|^2|_{p,S}(\lambda, \nu) \leq \int_{\nu_0}^{\nu} d\nu' e^{J \left(\int_{\nu_0}^{\nu} \mathcal{H} - \int_{\nu_0}^{\nu'} \mathcal{H} \right)} |r^{(J+2-\frac{2}{p})} \tilde{L}_X|_{p,S} \leq \int_{\nu_0}^{\nu} d\nu' e^{J \tilde{C}_0 \frac{\nu - \nu'}{\nu \nu'}} |r^{(J+2-\frac{2}{p})} \tilde{L}_X|_{p,S} . \quad (17.17)$$

Observing that

$$\nabla^J |X|^2 = \sum_{k=0}^J \binom{J}{k} \nabla^k |X| \nabla^{J-k} |X| = 2(\nabla^J |X|) |X| + \sum_{k=1}^{J-1} \binom{J}{k} \nabla^k |X| \nabla^{J-k} |X| ,$$

we write

$$\begin{aligned} & |r^{(J+2-\frac{2}{p})} \left(2(\nabla^J |X|)|X| + \sum_{k=1}^{J-1} \binom{J}{k} \nabla^k |X| \nabla^{J-k} |X| \right) |_{p,S}(\lambda, \nu) \\ & \leq \int_{\nu_0}^{\nu} d\nu' e^{J \left(\int_{\nu_0}^{\nu} \mathcal{H} - \int_{\nu_0}^{\nu'} \mathcal{H} \right)} |r^{(J+2-\frac{2}{p})} \tilde{L}_X|_{p,S} \leq \int_{\nu_0}^{\nu} d\nu' e^{J \tilde{C}_0 \frac{\nu-\nu'}{\nu\nu'}} |r^{(J+2-\frac{2}{p})} \tilde{L}_X|_{p,S} \quad (17.18) \end{aligned}$$

and

$$|r^{(J+2-\frac{2}{p})} \nabla^J |X| |X| |_{p,S}(\nu, \lambda) \leq \frac{1}{2} \sum_{k=1}^{J-1} \binom{J}{k} |r^{(J+2-\frac{2}{p})} \nabla^k |X| \nabla^{J-k} |X| |_{p,S}(\nu, \lambda) + \frac{1}{2} \int_{\nu_0}^{\nu} d\nu' e^{J \tilde{C}_0 \frac{\nu-\nu'}{\nu\nu'}} |r^{(J+2-\frac{2}{p})} \tilde{L}_X|_{p,S}(\nu', \lambda) .$$

We consider the sup of this inequality with respect to

$$\frac{r|X|}{\log r} \in [0, M \equiv \sup_{\nu} \left| \frac{r}{\log r} X \right|_{\infty, S}]$$

considering $|X|$ as an independent variable and observing that $M = O(\varepsilon)$, obtaining,

$$\sup |r^{(J+1-\frac{2}{p})} \log r \nabla^J |X| \left(\frac{r|X|}{\log r} \right) |_{p,S}(\nu, \lambda) \quad (17.19)$$

$$\leq \frac{1}{2} \sum_{k=1}^{J-1} \binom{J}{k} |r^{(J+2-\frac{2}{p})} \nabla^k |X| \nabla^{J-k} |X| |_{p,S}(\nu, \lambda) + \frac{1}{2} \sup \int_{\nu_0}^{\nu} d\nu' e^{J \tilde{C}_0 \frac{\nu-\nu'}{\nu\nu'}} |r^{(J+2-\frac{2}{p})} \tilde{L}_X|_{p,S}(\nu', \lambda) .$$

obtaining

$$\begin{aligned} \log r |r^{(J+1-\frac{2}{p})} \nabla^J |X| |_{p,S} & \leq M^{-1} \frac{1}{2} \sum_{k=1}^{J-1} \binom{J}{k} |r^{(J+2-\frac{2}{p})} \nabla^k |X| \nabla^{J-k} |X| |_{p,S} \\ & + M^{-1} \frac{1}{2} \int_{\nu_0}^{\nu} d\nu' e^{J \tilde{C}_0 \frac{\nu-\nu'}{\nu\nu'}} \left[M(\log r) |r^{(J+1-\frac{2}{p})} \nabla^J X \cdot \Omega \hat{\chi}|_{p,S} + |r^{(J+2-\frac{2}{p})} \nabla^J X \cdot \zeta|_{p,S} + \sup_{|X|} |r^{(J+2-\frac{2}{p})} \{good\}_X|_{p,S} \right] . \end{aligned}$$

We make the iterative assumption if ,

$$\left| \frac{r^{(J+1-\frac{2}{p})}}{\log r} \nabla^J |X| \right|_{p,S}(\lambda, \nu) \leq c \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\Gamma_1+\delta)}}{\rho^J} \right) , \quad (17.20)$$

with c of order $O(\epsilon)$ if $J \leq 7$ and of order $O(1)$ otherwise, where

$$\Gamma_1 = \Gamma_1(\nu) = C_1 \frac{\nu - \nu_0}{\nu \nu_0} . \quad (17.21)$$

To show the consistency of this assumption we proceed in the following way: first we consider the non integrated term

$$\frac{1}{2M} (\log r)^2 \sum_{k=1}^{J-1} \binom{J}{k} \left| \frac{r^{(J+2-\frac{2}{p})}}{(\log r)^2} \nabla^k |X| \nabla^{J-k} |X| \right|_{p,S}$$

$$\begin{aligned}
&\leq \frac{1}{2M}(\log r)^2 \sum_{k=1}^{\lfloor \frac{J-1}{2} \rfloor} \frac{J!}{(J-k)!k!} \left| \frac{r^{k+1}}{\log r} \nabla^k |X| \right|_{\infty, S} \left| \frac{r^{J-k+1-\frac{2}{p}}}{\log r} \nabla^{J-k} |X| \right|_{p, S} \\
&+ \frac{1}{2M}(\log r)^2 \sum_{k=\lfloor \frac{J-1}{2} \rfloor + 1}^{J-1} \frac{J!}{(J-k)!k!} \left| \frac{r^{k+1-\frac{2}{p}}}{\log r} \nabla^k |X| \right|_{p, S} \left| \frac{r^{J-k+1}}{\log r} \nabla^{J-k} |X| \right|_{\infty, S} \\
&\leq \frac{1}{M}(\log r)^2 \sum_{k=1}^{\lfloor \frac{J-1}{2} \rfloor} \frac{J!}{(J-k)!k!} \left| \frac{r^{k+1}}{\log r} \nabla^k |X| \right|_{\infty, S} \left| \frac{r^{J-k+1-\frac{2}{p}}}{\log r} \nabla^{J-k} |X| \right|_{p, S} \\
&\leq M^{-1} c (\log r)^2 \sum_{k=1}^{\lfloor \frac{J-1}{2} \rfloor} \frac{J!}{(J-k)!k!} \frac{(k+1)!}{(k+1)^\alpha} \frac{e^{(k-1)(\Gamma_1+\delta)}}{\rho^{k+1}} \frac{(J-k)!}{(J-k)^\alpha} \frac{e^{(J-k-2)(\Gamma_1+\delta)}}{\rho^{J-k}} \\
&\leq c (\log r)^2 \frac{J!}{J^\alpha} \frac{e^{(J-2)(\Gamma_1+\delta_1)}}{\rho^J} \left(M^{-1} \frac{e^{-(\Gamma_1+\delta)}}{\rho} \right) \sum_{k=1}^{\lfloor \frac{J-1}{2} \rfloor} \frac{1!}{(k+1)^{\alpha-1}} \frac{J^\alpha}{(J-k)^\alpha} \\
&\leq (\log r)^2 \frac{1}{100} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\Gamma_1+\delta)}}{\rho^J} \right), \tag{17.22}
\end{aligned}$$

choosing δ sufficiently large and $\alpha > 2$. Let us control the delicate integral parts,

$$\begin{aligned}
&\int_{\nu_0}^{\nu} d\nu' e^{J\tilde{C}_0 \frac{\nu-\nu'}{\nu\nu'}} |(\log r) r^{(J+1-\frac{2}{p})} \nabla^J X \cdot \Omega \hat{\chi}|_{p, S} \leq \int_{\nu_0}^{\nu} d\nu' e^{J\tilde{C}_0 \frac{\nu-\nu'}{\nu\nu'}} (\log r)^2 \left| \frac{r^{(J+1-\frac{2}{p})}}{\log r} \nabla^J X \right|_{p, S} |\Omega \hat{\chi}|_{\infty, S} \\
&\leq c\varepsilon_0 \int_{\nu_0}^{\nu} \frac{d\nu' (\log \nu')^2}{\nu'^2} e^{J\tilde{C}_0 \frac{\nu-\nu'}{\nu\nu'}} \left| \frac{r^{(J+1-\frac{2}{p})}}{\log r} \nabla^J X \right|_{p, S} \tag{17.23} \\
&\leq c\varepsilon_0 \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\Gamma_1(\lambda, \nu)+\delta)}}{\rho^J} \right) \int_{\nu_0}^{\nu} \frac{d\nu' (\log \nu')^2}{\nu'^2} e^{[J\tilde{C}_0 \frac{\nu-\nu'}{\nu\nu'} - (J-2)(\Gamma_1(\nu)-\Gamma_1(\nu'))]} \\
&\leq c\varepsilon_0 \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\Gamma_1+\delta_1)}}{\rho^J} \right),
\end{aligned}$$

provided $C_1 > \tilde{C}_0$. The second integral term is more delicate

$$\begin{aligned}
&\int_{\nu_0}^{\nu} d\nu' e^{J\tilde{C}_0 \frac{\nu-\nu'}{\nu\nu'}} \left| r^{(J+2-\frac{2}{p})} \nabla^J X \cdot \zeta \right|_{p, S} \leq \int_{\nu_0}^{\nu} d\nu' e^{J\tilde{C}_0 \frac{\nu-\nu'}{\nu\nu'}} (\log r) \left| \frac{r^{(J+1-\frac{2}{p})}}{\log r} \nabla^J X \right|_{p, S} |r\zeta|_{\infty, S} \\
&\leq c\varepsilon_0 \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\Gamma_1(\lambda, \nu)+\delta)}}{\rho^J} \right) (\log r) (\lambda, \nu) \int_{\nu_0}^{\nu} \frac{d\nu'}{\nu'} e^{[J\tilde{C}_0 \frac{\nu-\nu'}{\nu\nu'} - (J-2)(\Gamma_1(\nu)-\Gamma_1(\nu'))]} \\
&\leq c(\log r)^2 \varepsilon_0 \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\Gamma_1(\nu)+\delta_1)}}{\rho^J} \right). \tag{17.24}
\end{aligned}$$

Therefore

$$M^{-1} \frac{1}{2} \int_{\nu_0}^{\nu} d\nu' e^{J\tilde{C}_0 \frac{\nu-\nu'}{\nu\nu'}} \left| \frac{r^{(J+2-\frac{2}{p})}}{\log r} \nabla^J X \cdot \zeta \right|_{p, S} \leq \frac{c}{2} (\log r)^2 \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\Gamma_1(\nu)+\delta_1)}}{\rho^J} \right). \tag{17.25}$$

The correction terms can be treated as before, the only request they imply is that $\delta_1 \geq \delta$. They contribute as

$$\frac{1}{100}c(\log r)^2 \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\Gamma_1(\lambda, \nu) + \delta_1)}}{\rho^J} \right).$$

Collecting all the estimates together we conclude that

$$\log r |r^{(J+1-\frac{2}{p})} \nabla^J |X||_{p,S} \leq (\log r)^2 \frac{1}{100} \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\Gamma_1 + \delta)}}{\rho^J} \right) + \quad (17.26)$$

$$+ \frac{c}{2}(\log r)^2 \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\Gamma_1(\nu) + \delta_1)}}{\rho^J} \right) + \frac{1}{100}c(\log r)^2 \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\Gamma_1(\nu) + \delta_1)}}{\rho^J} \right) \\ \leq c(\log r)^2 \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\Gamma_1(\nu) + \delta_1)}}{\rho^J} \right) \quad (17.27)$$

which implies the inequality 11.15 for all J ,¹³⁵

$$|\frac{r^{(J+1-\frac{2}{p})}}{\log r} \nabla^J |X||_{p,S}(\lambda, \nu) \leq c \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\Gamma_1 + \delta)}}{\rho^J} \right).$$

18 Appendix to Section 13

18.1 Proof of Lemma 13.1

To prove this lemma we derive first the corresponding equations for the norms $|r^{2-\frac{2}{p}}\zeta|_{p,S}$ and $||\lambda|^2 r^{\frac{3}{2}+\epsilon-\frac{2}{p}}\underline{\beta}|_{p,S}$; they are obtained in the following way, denoting $L_1 = [\mathcal{D}_\lambda \nabla \log \Omega]$,

$$\begin{aligned} \frac{\partial}{\partial \lambda} |r^{2-\frac{2}{p}}\zeta|^p &= p|r^{2-\frac{2}{p}}\zeta|^{p-2} (r^{2-\frac{2}{p}}\zeta) \cdot \frac{\partial}{\partial \lambda} (r^{2-\frac{2}{p}}\zeta) \\ &= p|r^{2-\frac{2}{p}}\zeta|^{p-2} (r^{2-\frac{2}{p}}\zeta) \cdot \left\{ -r^{2-\frac{2}{p}} (\Omega \text{tr} \underline{\chi} \zeta + 2\Omega \underline{\hat{\chi}} \cdot \zeta + \Omega \underline{\beta} + L_1) + (2 - \frac{2}{p}) r^{1-\frac{2}{p}} \frac{\partial r}{\partial \lambda} \zeta \right\} \\ &= p|r^{2-\frac{2}{p}}\zeta|^{p-2} (r^{2-\frac{2}{p}}\zeta) \cdot \left\{ -r^{2-\frac{2}{p}} (\Omega \text{tr} \underline{\chi} \zeta + 2\Omega \underline{\hat{\chi}} \cdot \zeta + \Omega \underline{\beta} + L_1) + (2 - \frac{2}{p}) r^{2-\frac{2}{p}} \frac{1}{2} \overline{\Omega \text{tr} \underline{\chi}} \zeta \right\} \\ &= p|r^{2-\frac{2}{p}}\zeta|^{p-2} \left\{ \left[-(\Omega \text{tr} \underline{\chi}) |r^{2-\frac{2}{p}}\zeta|^2 - 2\Omega (r^{2-\frac{2}{p}}\zeta) \cdot \underline{\hat{\chi}} \cdot (r^{2-\frac{2}{p}}\zeta) - r^{2(2-\frac{2}{p})} \Omega \underline{\beta} \cdot \zeta - r^{2(2-\frac{2}{p})} L_1 \cdot \zeta \right] + (1 - \frac{1}{p}) \overline{\Omega \text{tr} \underline{\chi}} |r^{2-\frac{2}{p}}\zeta|^2 \right\} \\ &= (-p\Omega \text{tr} \underline{\chi} + p\overline{\Omega \text{tr} \underline{\chi}} - \overline{\Omega \text{tr} \underline{\chi}}) |r^{2-\frac{2}{p}}\zeta|^p - p2\Omega (r^{2-\frac{2}{p}}\zeta) \cdot \underline{\hat{\chi}} \cdot (r^{2-\frac{2}{p}}\zeta) |r^{2-\frac{2}{p}}\zeta|^{p-2} - p|r^{2-\frac{2}{p}}\zeta|^{p-2} r^{2(2-\frac{2}{p})} \Omega \underline{\beta} \cdot \zeta \\ &\quad - p|r^{2-\frac{2}{p}}\zeta|^{p-2} r^{2(2-\frac{2}{p})} L_1 \cdot \zeta \end{aligned}$$

¹³⁵With some extra work it is possible to prove that

$$|\frac{r^{J-\frac{2}{p}}}{\log r} \partial^J X_a|_{p,S} \leq c\varepsilon_0 \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\Gamma_1 + \delta)}}{\rho^J} \right).$$

$$= ((p-1)(\overline{\Omega \text{tr} \underline{\chi}} - \Omega \text{tr} \underline{\chi}) - (\Omega \text{tr} \underline{\chi})) |r^{2-\frac{2}{p}} \underline{\zeta}|^p - 2p\Omega(r^{2-\frac{2}{p}} \underline{\zeta}) \cdot \hat{\underline{\chi}} \cdot (r^{2-\frac{2}{p}} \underline{\zeta}) |r^{2-\frac{2}{p}} \underline{\zeta}|^{p-2} - p|r^{2-\frac{2}{p}} \underline{\zeta}|^{p-2} r^{2(2-\frac{2}{p})} \Omega \underline{\beta} \cdot \underline{\zeta} - p|r^{2-\frac{2}{p}} \underline{\zeta}|^{p-2} r^{2(2-\frac{2}{p})} L_1 \cdot \underline{\zeta} . \quad (18.1)$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial \lambda} |r^{2-\frac{2}{p}} \underline{\zeta}|_{p,S}^p &= \frac{\partial}{\partial \lambda} \int_S |r^{2-\frac{2}{p}} \underline{\zeta}|^p = \int_S \left(\frac{\partial}{\partial \lambda} |r^{2-\frac{2}{p}} \underline{\zeta}|^p + \Omega \text{tr} \underline{\chi} |r^{2-\frac{2}{p}} \underline{\zeta}|^p \right) \\ &= -(p-1) \int_S (\Omega \text{tr} \underline{\chi} - \overline{\Omega \text{tr} \underline{\chi}}) |r^{2-\frac{2}{p}} \underline{\zeta}|^p - p \int_S 2\Omega(r^{2-\frac{2}{p}} \underline{\zeta}) \cdot \hat{\underline{\chi}} \cdot (r^{2-\frac{2}{p}} \underline{\zeta}) |r^{2-\frac{2}{p}} \underline{\zeta}|^{p-2} + |r^{2-\frac{2}{p}} \underline{\zeta}|^{p-2} r^{2(2-\frac{2}{p})} \Omega \underline{\beta} \cdot \underline{\zeta} \\ &\leq p \int_S (|\Omega \text{tr} \underline{\chi} - \overline{\Omega \text{tr} \underline{\chi}}| + |2\Omega \hat{\underline{\chi}}|) |r^{2-\frac{2}{p}} \underline{\zeta}| |r^{2-\frac{2}{p}} \underline{\zeta}|^{p-1} + p \int_S \left(|r^{(2-\frac{2}{p})} \Omega \underline{\beta}| |r^{2-\frac{2}{p}} \underline{\zeta}|^{p-1} + |r^{(2-\frac{2}{p})} L_1| |r^{2-\frac{2}{p}} \underline{\zeta}|^{p-1} \right) . \end{aligned} \quad (18.2)$$

Applying Hoder inequality to the r.h.s. we obtain

$$\begin{aligned} \frac{\partial}{\partial \lambda} |r^{2-\frac{2}{p}} \underline{\zeta}|_{p,S}^p &\leq (|\Omega \text{tr} \underline{\chi} - \overline{\Omega \text{tr} \underline{\chi}}| + |2\Omega \hat{\underline{\chi}}|) |r^{2-\frac{2}{p}} \underline{\zeta}|_{p,S} p |r^{2-\frac{2}{p}} \underline{\zeta}|_{p,S}^{p-1} \\ &\quad + |r^{2-\frac{2}{p}} \Omega \underline{\beta}|_{p,S} p |r^{2-\frac{2}{p}} \underline{\zeta}|_{p,S}^{p-1} + |r^{2-\frac{2}{p}} L_1|_{p,S} p |r^{2-\frac{2}{p}} \underline{\zeta}|_{p,S}^{p-1} \end{aligned} \quad (18.3)$$

and from it

$$\frac{\partial}{\partial \lambda} |r^{2-\frac{2}{p}} \underline{\zeta}|_{p,S} \leq (|\Omega \text{tr} \underline{\chi} - \overline{\Omega \text{tr} \underline{\chi}}|_{\infty,S} + |2\Omega \hat{\underline{\chi}}|_{\infty,S}) |r^{2-\frac{2}{p}} \underline{\zeta}|_{p,S} + |r^{2-\frac{2}{p}} \Omega \underline{\beta}|_{p,S} + |r^{2-\frac{2}{p}} L_1|_{p,S} . \quad (18.4)$$

Let us obtain the differential inequality for $\underline{\beta}$, with $L_2 = -[\Omega(\text{div} \underline{\alpha} + \nabla \log \Omega \cdot \underline{\alpha})]$,

$$\mathcal{D}_\lambda \underline{\beta} + 2\Omega \text{tr} \underline{\chi} \underline{\beta} + 2\Omega \underline{\omega} \underline{\beta} - \Omega \underline{\alpha} \cdot \underline{\zeta} - [\Omega(\text{div} \underline{\alpha} + \nabla \log \Omega \cdot \underline{\alpha})] = 0$$

$$\begin{aligned} \frac{\partial}{\partial \lambda} |r^{4-\frac{2}{p}} \underline{\beta}|^p &= p |r^{4-\frac{2}{p}} \underline{\beta}|^{p-2} (r^{4-\frac{2}{p}} \underline{\beta}) \cdot \frac{\partial}{\partial \lambda} (r^{4-\frac{2}{p}} \underline{\beta}) \\ &= p |r^{4-\frac{2}{p}} \underline{\beta}|^{p-2} (r^{4-\frac{2}{p}} \underline{\beta}) \cdot \left\{ -r^{4-\frac{2}{p}} (2\Omega \text{tr} \underline{\chi} \underline{\beta} + 2\Omega \underline{\omega} \underline{\beta} - \Omega \underline{\zeta} \cdot \underline{\alpha} + L_2) + (4 - \frac{2}{p}) r^{3-\frac{2}{p}} \frac{\partial r}{\partial \lambda} \underline{\beta} \right\} \\ &= p |r^{4-\frac{2}{p}} \underline{\beta}|^{p-2} (r^{4-\frac{2}{p}} \underline{\beta}) \cdot \left\{ -r^{4-\frac{2}{p}} (2\Omega \text{tr} \underline{\chi} \underline{\beta} + 2\Omega \underline{\omega} \underline{\beta} - \Omega \underline{\zeta} \cdot \underline{\alpha} + L_2) + (4 - \frac{2}{p}) r^{4-\frac{2}{p}} \frac{1}{2} \overline{\Omega \text{tr} \underline{\chi}} \underline{\beta} \right\} \\ &= p |r^{4-\frac{2}{p}} \underline{\beta}|^{p-2} \left\{ \left[-(2\Omega \text{tr} \underline{\chi}) |r^{4-\frac{2}{p}} \underline{\beta}|^2 - 2\Omega \underline{\omega} |r^{4-\frac{2}{p}} \underline{\beta}|^2 + r^{2(4-\frac{2}{p})} \Omega \underline{\zeta} \cdot \underline{\alpha} \cdot \underline{\beta} - r^{2(4-\frac{2}{p})} L_2 \cdot \underline{\beta} \right] \right. \\ &\quad \left. + (2 - \frac{1}{p}) \overline{\Omega \text{tr} \underline{\chi}} |r^{4-\frac{2}{p}} \underline{\beta}|^2 \right\} \\ &= ((2p-1)(\overline{\Omega \text{tr} \underline{\chi}} - \Omega \text{tr} \underline{\chi}) - (\Omega \text{tr} \underline{\chi})) |r^{4-\frac{2}{p}} \underline{\beta}|^p - 2p\Omega \underline{\omega} |r^{4-\frac{2}{p}} \underline{\beta}|^p + p |r^{4-\frac{2}{p}} \underline{\beta}|^{p-2} r^{2(4-\frac{2}{p})} \Omega \underline{\zeta} \cdot \underline{\alpha} \cdot \underline{\beta} \\ &\quad - p |r^{4-\frac{2}{p}} \underline{\beta}|^{p-2} r^{2(4-\frac{2}{p})} L_2 \cdot \underline{\beta} . \end{aligned} \quad (18.5)$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial \lambda} |r^{4-\frac{2}{p}} \underline{\beta}|_{p,S}^p &= \frac{\partial}{\partial \lambda} \int_S |r^{4-\frac{2}{p}} \underline{\beta}|^p = \int_S \left(\frac{\partial}{\partial \lambda} |r^{4-\frac{2}{p}} \underline{\beta}|^p + \Omega \text{tr} \underline{\chi} |r^{4-\frac{2}{p}} \underline{\beta}|^p \right) \\ &= -(2p-1) \int_S (\Omega \text{tr} \underline{\chi} - \overline{\Omega \text{tr} \underline{\chi}}) |r^{4-\frac{2}{p}} \underline{\beta}|^p - p \int_S 2\Omega \underline{\omega} |r^{4-\frac{2}{p}} \underline{\beta}|^p - |r^{4-\frac{2}{p}} \underline{\beta}|^{p-2} r^{2(4-\frac{2}{p})} \Omega \underline{\zeta} \cdot \underline{\alpha} \cdot \underline{\beta} + |r^{4-\frac{2}{p}} \underline{\beta}|^{p-2} r^{2(4-\frac{2}{p})} L_2 \cdot \underline{\beta} \\ &\leq 2p \int_S (|\Omega \text{tr} \underline{\chi} - \overline{\Omega \text{tr} \underline{\chi}}| + |2\Omega \underline{\omega}|) |r^{4-\frac{2}{p}} \underline{\beta}| |r^{4-\frac{2}{p}} \underline{\beta}|^{p-1} + p \int_S \left(|r^{(4-\frac{2}{p})} \Omega \underline{\zeta} \cdot \underline{\alpha}| |r^{4-\frac{2}{p}} \underline{\beta}|^{p-1} + |r^{(4-\frac{2}{p})} L_2| |r^{4-\frac{2}{p}} \underline{\beta}|^{p-1} \right) . \end{aligned}$$

Applying Hoder inequality to the r.h.s. we obtain

$$\begin{aligned} \frac{\partial}{\partial \lambda} |r^{4-\frac{2}{p}} \underline{\beta}|_{p,S}^p &\leq (|\Omega \text{tr} \underline{\chi} - \overline{\Omega \text{tr} \underline{\chi}}| + |2\Omega \underline{\omega}|) |r^{4-\frac{2}{p}} \underline{\beta}|_{p,S} 2p |r^{4-\frac{2}{p}} \underline{\beta}|_{p,S}^{p-1} \\ &\quad + |r^{4-\frac{2}{p}} \Omega \zeta \cdot \underline{\alpha}|_{p,S} p |r^{4-\frac{2}{p}} \underline{\beta}|_{p,S}^{p-1} + |r^{4-\frac{2}{p}} L_2|_{p,S} p |r^{4-\frac{2}{p}} \underline{\beta}|_{p,S}^{p-1} \end{aligned} \quad (18.6)$$

and from it

$$\frac{\partial}{\partial \lambda} |r^{4-\frac{2}{p}} \underline{\beta}|_{p,S} \leq (2|\Omega \text{tr} \underline{\chi} - \overline{\Omega \text{tr} \underline{\chi}}|_{\infty,S} + |4\Omega \underline{\omega}|_{\infty,S}) |r^{4-\frac{2}{p}} \underline{\beta}|_{p,S} + |r^{4-\frac{2}{p}} \Omega \zeta \cdot \underline{\alpha}|_{p,S} + |r^{4-\frac{2}{p}} L_2|_{p,S} . \quad (18.7)$$

Collecting the results we have the following p.d.e. inequalities,

$$\begin{aligned} \frac{\partial}{\partial \lambda} |r^{2-\frac{2}{p}} \zeta|_{p,S} &\leq (|\Omega \text{tr} \underline{\chi} - \overline{\Omega \text{tr} \underline{\chi}}|_{\infty,S} + |2\Omega \underline{\chi}|_{\infty,S}) |r^{2-\frac{2}{p}} \zeta|_{p,S} + |r^{2-\frac{2}{p}} \Omega \underline{\beta}|_{p,S} + |r^{2-\frac{2}{p}} L_1|_{p,S} \quad (18.8) \\ \frac{\partial}{\partial \lambda} |r^{4-\frac{2}{p}} \underline{\beta}|_{p,S} &\leq (2|\Omega \text{tr} \underline{\chi} - \overline{\Omega \text{tr} \underline{\chi}}|_{\infty,S} + |4\Omega \underline{\omega}|_{\infty,S}) |r^{4-\frac{2}{p}} \underline{\beta}|_{p,S} + |r^{4-\frac{2}{p}} \Omega \zeta \cdot \underline{\alpha}|_{p,S} + |r^{4-\frac{2}{p}} L_2|_{p,S} . \end{aligned}$$

Integrating along \underline{C}_0 from $S_0 = S(\lambda_0, \nu_0)$ to $S(\lambda, \nu_0)$ we obtain

$$\begin{aligned} |r^{2-\frac{2}{p}} \zeta|_{p,S}(\lambda, \nu_0) &\leq c \left(|r^{2-\frac{2}{p}} \zeta|_{p,S}(\lambda_0, \nu_0) + \int_{\lambda_0}^{\lambda} d\lambda' \left(|r^{2-\frac{2}{p}} \Omega \underline{\beta}|_{p,S} + |r^{2-\frac{2}{p}} L_1|_{p,S} \right)(\lambda', \nu_0) \right) \quad (18.9) \\ |r^{4-\frac{2}{p}} \underline{\beta}|_{p,S}(\lambda, \nu_0) &\leq c \left(|r^{4-\frac{2}{p}} \underline{\beta}|_{p,S}(\lambda_0, \nu_0) + \int_{\lambda_0}^{\lambda} d\lambda' \left(|r^{4-\frac{2}{p}} \Omega \zeta \cdot \underline{\alpha}|_{p,S} + |r^{4-\frac{2}{p}} L_2|_{p,S} \right)(\lambda', \nu_0) \right) . \end{aligned}$$

Before getting the estimates for ζ and $\underline{\beta}$ we modify the second one multiplying both sides with $|\lambda|^{\frac{3}{2}+\delta}/r^{2-\epsilon}$, with $\epsilon > 0, \delta \geq 0$ which we are allowed to do as this function is increasing in r if $\nu_0 \leq 2r$. In fact, assume $|\lambda| = 2r - \nu_0$ on \underline{C}_0 if $r \geq \nu_0/2$ which implies on \underline{C}_0 , $r \geq t$, therefore points outside the null cone with vertex in the origin (in Minkowski spacetime). It follows

$$\begin{aligned} \frac{d}{dr} \frac{|\lambda|^{\frac{3}{2}+\delta}}{r^{2-\epsilon}} &= 2(3/2 + \delta) \frac{(2r - \nu_0)^{\frac{1}{2}+\delta}}{r^{2-\epsilon}} - (2 - \epsilon) \frac{(2r - \nu_0)^{\frac{3}{2}+\delta}}{r^{3-\epsilon}} \quad (18.10) \\ &= (2 - \epsilon) \frac{(2r - \nu_0)^{\frac{1}{2}+\delta}}{r^{2-\epsilon}} \left[\frac{2(3/2 + \delta)}{2 - \epsilon} - \frac{(2r - \nu_0)}{r} \right] \\ &= (2 - \epsilon) \frac{(2r - \nu_0)^{\frac{1}{2}+\delta}}{r^{2-\epsilon}} \left[\frac{2(3/2 + \delta)}{2 - \epsilon} - \frac{r - t}{r} \right] \geq (2 - \epsilon) \frac{(2r - \nu_0)^{\frac{1}{2}+\delta}}{r^{2-\epsilon}} [(3/2 + \delta) - 1] \geq 0 . \end{aligned}$$

Therefore

$$\begin{aligned} |r^{2-\frac{2}{p}} \zeta|_{p,S}(\lambda, \nu_0) &\leq c \left(|r^{2-\frac{2}{p}} \zeta|_{p,S}(\lambda_0, \nu_0) + \int_{\lambda_0}^{\lambda} d\lambda' \left(|r^{2-\frac{2}{p}} \Omega \underline{\beta}|_{p,S} + |r^{2-\frac{2}{p}} L_1|_{p,S} \right)(\lambda', \nu_0) \right) \quad (18.11) \\ ||\lambda|^{\frac{3}{2}+\delta} r^{2+\epsilon-\frac{2}{p}} \underline{\beta}|_{p,S}(\lambda, \nu_0) &\leq c \left(||\lambda|^{\frac{3}{2}+\delta} r^{2+\epsilon-\frac{2}{p}} \underline{\beta}|_{p,S}(\lambda_0, \nu_0) + \int_{\lambda_0}^{\lambda} d\lambda' \left(||\lambda|^{\frac{3}{2}+\delta} r^{2+\epsilon-\frac{2}{p}} \Omega \zeta \cdot \underline{\alpha}|_{p,S} + ||\lambda|^{\frac{3}{2}+\delta} r^{2+\epsilon-\frac{2}{p}} L_2|_{p,S} \right) \right) . \end{aligned}$$

the integral estimates 18.11 to be meaningful require that the integrals are bounded, which happens if we require that on \underline{C}_0 , with $\tilde{\delta} > \delta \geq 0$,

$$\mathcal{D}_\lambda \nabla \Omega = O\left(r^{-(2+\epsilon)}|\lambda|^{-(1+\tilde{\delta})}\right) \quad , \quad \underline{\alpha} = O\left(r^{-(1+\epsilon)}|\lambda|^{-(\frac{5}{2}+\tilde{\delta})}\right). \quad (18.12)$$

and prove that $\underline{\beta} = O\left(r^{-(2+\epsilon)}|\lambda|^{-\frac{3}{2}}\right)$, $\zeta = O(r^{-2})$. We can choose $\delta = 0$ and the two integral inequalities become

$$\begin{aligned} |r^{2+\epsilon-\frac{2}{p}}\zeta|_{p,S}(\lambda, \nu_0) &\leq c \left(|r^{2+\epsilon-\frac{2}{p}}\zeta|_{p,S}(\lambda_0, \nu_0) + \int_{\lambda_0}^{\lambda} d\lambda' \left(|r^{2+\epsilon-\frac{2}{p}}\Omega \underline{\beta}|_{p,S} + |r^{2+\epsilon-\frac{2}{p}}L_1|_{p,S} \right) \right) \\ ||\lambda|^{\frac{3}{2}}r^{2+\epsilon-\frac{2}{p}}\underline{\beta}|_{p,S}(\lambda, \nu_0) &\leq c \left(||\lambda|^{\frac{3}{2}}r^{2+\epsilon-\frac{2}{p}}\underline{\beta}|_{p,S}(\lambda_0, \nu_0) + \int_{\lambda_0}^{\lambda} d\lambda' \left(||\lambda|^{\frac{3}{2}}r^{2+\epsilon-\frac{2}{p}}\Omega \zeta \cdot \underline{\alpha}|_{p,S} + ||\lambda|^{\frac{3}{2}}r^{2+\epsilon-\frac{2}{p}}L_2|_{p,S} \right) \right). \end{aligned} \quad (18.13)$$

It is easy to prove that the expected result holds, namely

$$\underline{\beta} = O\left(r^{-(2+\epsilon)}|\lambda|^{-\frac{3}{2}}\right) \quad , \quad \zeta = O(r^{-2}) \quad ,$$

assuming 18.12 and moreover that on S_0 we have

$$|r^{2-\frac{2}{p}}\zeta|_{p,S}(\lambda_0, \nu_0) \leq c\varepsilon \quad , \quad ||\lambda|^{\frac{3}{2}}r^{2+\epsilon-\frac{2}{p}}\underline{\beta}|_{p,S}(\lambda_0, \nu_0) \leq c\varepsilon. \quad (18.14)$$

Remarks:

i) It is clear that this estimate has to be generalized to all the ∇ derivatives of ζ and $\underline{\beta}$ with the standard mechanism we used to prove the estimates for the ∇ derivatives in the interior. The way to proceed should be again an inductive one, we assume the desired estimates for $\nabla^J \zeta$ and $\nabla^J \underline{\beta}$, with $J < N$ and prove the same estimates for $J = N$. This implies a correct choice of $\underline{\Gamma}_0$.

ii) Observe that the way to estimate ζ on \underline{C}_0 is different from the one used on C_0 . Here the only loss of derivatives is due to the presence of the term $\underline{\beta}$.

18.1.1 The initial data on $S_0 = S(\lambda_0, \nu_0)$

From the previous considerations concerning the estimates of the quantities on \underline{C}_0 it follows that the following behaviours have to be required on S_0 , to satisfy conditions 7.2

i) From the estimates on \underline{C}_0 , $J > 0$:

$$\begin{aligned} |r^{2+J-\frac{2}{p}}\nabla^J \zeta|_{p,S}(\lambda_0, \nu_0) &\leq c_1 \varepsilon \left(\frac{(J+1)!}{(J+1)^\alpha} \frac{e^{(J-2)(\delta_0+\Gamma_0)}}{\rho_{0,0}^J} \right) \\ ||\lambda|^{\frac{3}{2}}r^{2+\epsilon-\frac{2}{p}}\nabla^J \underline{\beta}|_{p,S}(\lambda_0, \nu_0) &\leq c_2 \varepsilon \left(\frac{(J+1)!}{(J+1)^\alpha} \frac{e^{(J-2)(\delta_0+\Gamma_0)}}{\rho_{0,0}^J} \right) \\ |r^{2+J-\frac{2}{p}}\nabla^J \text{tr} \underline{\chi}|_{p,S}(\lambda_0, \nu_0) &\leq c_3 \varepsilon \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0+\Gamma_{0,0})}}{\rho_0^J} \right) \\ ||\lambda|^{\frac{3}{2}}r^{1+J+\epsilon-\frac{2}{p}}\nabla^J \underline{\chi}|_{p,S}(\lambda_0, \nu_0) &\leq c_4 \varepsilon \left(\frac{J!}{J^\alpha} \frac{e^{(J-2)(\delta_0+\Gamma_0)}}{\rho_{0,0}^J} \right), \end{aligned} \quad (18.15)$$

together with the requirements that on S_0

$$\nabla \text{tr} \underline{\chi} - \zeta \text{tr} \underline{\chi} = O\left(\varepsilon |\lambda_0|^{\frac{3}{2}+\epsilon} r_0^2\right). \quad (18.16)$$

Remarks:

i) The first two conditions of 18.15 are required to obtain the estimates for $\nabla^J \zeta$ on \underline{C}_0 . The last condition, 18.16, is needed to have $\underline{\beta}$ with the right estimates on \underline{C}_0 , this is, basically a repetition of what has been done for β on C_0 . To clarify this point let us recall the expression of $\underline{\beta}$

$$\underline{\beta} = -(\nabla \text{tr} \underline{\chi} - \text{div} \underline{\chi} - \zeta \text{tr} \underline{\chi} + \zeta \cdot \underline{\chi}),$$

and from it the following one

$$\nabla^J \underline{\beta} = -\frac{1}{2} \nabla^{J+1} \text{tr} \underline{\chi} + \nabla^J \text{div} \hat{\chi} + \frac{1}{2} \nabla^J (\zeta \text{tr} \underline{\chi}) - \nabla^J (\zeta \cdot \hat{\chi}). \quad (18.17)$$

From the requirement that the l.h.s. decays as $O\left(|\lambda|^{\frac{3}{2}} r^{2+\epsilon}\right)$ it follows that this is automatically satisfied by $\nabla^J \text{div} \hat{\chi}$ and by $\nabla^J (\zeta \cdot \hat{\chi})$ while it is not from the terms, considered separately, $-\frac{1}{2} \nabla^{J+1} \text{tr} \underline{\chi}$ and $\frac{1}{2} \nabla^J (\zeta \text{tr} \underline{\chi})$; to fulfill the condition we have, therefore, to require condition 18.16.

ii) Notationally we recall that the various ρ 's used in the initial data estimates satisfy the following inequalities

$$\rho_0 < \rho_{0,0,1} < \rho_{0,0}, \quad (18.18)$$

where ρ_0 will be used when we obtain the internal estimates.

The condition for $\underline{\omega}$ on S_0 :

We start from

$$\frac{d}{d\nu} \underline{\omega} - 2\Omega \omega \underline{\omega} = F \quad (18.19)$$

where

$$F = \zeta \cdot \nabla \log \Omega + \frac{3}{2} |\zeta|^2 - \frac{1}{2} |\nabla \log \Omega|^2 - \frac{1}{2} (\mathbf{K} + \frac{1}{4} \text{tr} \chi \text{tr} \underline{\chi} - \frac{1}{2} \hat{\chi} \cdot \hat{\chi}) \quad (18.20)$$

Let us observe that

$$\frac{d}{d\nu} \left(e^{\int_{\nu_0}^{\nu} (-2\Omega \omega) d\nu'} \underline{\omega} \right) = e^{\int_{\nu_0}^{\nu} (-2\Omega \omega) d\nu'} \left[-2\Omega \omega \underline{\omega} + \frac{d}{d\nu} \underline{\omega} \right] = e^{\int_{\nu_0}^{\nu} (-2\Omega \omega) d\nu'} F(\nu). \quad (18.21)$$

Integrating

$$e^{\int_{\nu_0}^{\nu} (-2\Omega \omega) d\nu'} \underline{\omega}(\nu) - \underline{\omega}(\nu_0) = \int_{\nu_0}^{\nu} d\nu' e^{\int_{\nu_0}^{\nu'} (-2\Omega \omega) d\nu''} F(\nu') \quad (18.22)$$

Therefore

$$\underline{\omega}(\nu) = e^{\int_{\nu_0}^{\nu} (2\Omega\omega) d\nu'} \underline{\omega}(\nu_0) + e^{\int_{\nu_0}^{\nu} (2\Omega\omega) d\nu'} \int_{\nu_0}^{\nu} d\nu' e^{\int_{\nu_0}^{\nu'} (-2\Omega\omega) d\nu''} F(\nu') \quad (18.23)$$

and to require the $\lim_{\nu \rightarrow \infty} \underline{\omega}(\nu) = 0$ we need that

$$e^{\int_{\nu_0}^{\infty} (2\Omega\omega) d\nu'} \left[\underline{\omega}(\nu_0) + \int_{\nu_0}^{\infty} d\nu' e^{\int_{\nu_0}^{\nu'} (-2\Omega\omega) d\nu''} F(\nu') \right] = 0 \quad (18.24)$$

which implies that

$$\begin{aligned} \underline{\omega}(\nu_0) &= - \int_{\nu_0}^{\infty} d\nu' e^{\int_{\nu_0}^{\nu'} (-2\Omega\omega) d\nu''} F(\nu') = \\ &= - \int_{\nu_0}^{\infty} d\nu' e^{\int_{\nu_0}^{\nu'} (-2\Omega\omega) d\nu''} \left[\zeta \cdot \nabla \log \Omega + \frac{3}{2} |\zeta|^2 - \frac{1}{2} |\nabla \log \Omega|^2 - \frac{1}{2} (\mathbf{K} + \frac{1}{4} \text{tr} \chi \text{tr} \underline{\chi} - \frac{1}{2} \hat{\chi} \cdot \hat{\underline{\chi}}) \cdot \right] (\nu'). \end{aligned}$$

In conclusion what we proved in the previous subsections complete the proof of Theorem 3.2.

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